STRUCTURAL DYNAMIC ANALYSIS OF ROTOR BLADES WITH SWEPT TIPS

by

M. VENU GOPAL

DEPARTMENT OF AEROSPACE ENGINEERING

INDIAN INSTITUTE OF TECHNOLOGY KANPUR

May, 1999

STRUCTURAL DYNAMIC ANALYSIS OF ROTOR BLADES WITH SWEPT TIPS

A Thesis Submitted

in Partial Fulfilment of the Requirements

for the Degree of

MASTER OF TECHNOLOGY

by

M.VENU GOPAL

to the

DEPARTMENT OF AEROSPACE ENGINEERING

INDIAN INSTITUTE OF TECHNOLOGY KANPUR

MAY, 1999

CERTIFICATE

It is certified that the work contained in the thesis entitled "**STRUCTURAL DYNAMIC ANALYSIS OF ROTOR BLADE WITH SWEPT TIPS** ", by **M.VENU GOPAL** has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

Dr. C.VENKATESAN
Associate Professor
Dept. of Aerospace Engineering
I.I.T. Kanpur

May, 1999

ACKNOWLEDGEMENTS

I take this opportunity to thank my thesis supervisor Dr. C.Venkatesan , for his inspiration and valuable suggestions in carrying out this work. I consider myself fortunate to have worked under him for the freedom and constant encouragement he gave throughout this work.

I express my special thanks to my friends ValiBabu, Kaliprasadanna, Ravianna, Ramakoti, Vikky, Vinay, Narsi, Lala, Santhosh, Raju, Bhargav, Gopal, Janardhan, Sharath, Giri, $Chandu^2$, Chadra sekhar, Kkis, S.B.Raoanna, Rajananna,Prasanth for their tremendous help in bringing this work to the present form.

I would like to thank Mr. Rameshwar, Siva, Anil, Sada and many other friends for their suggestions during my course of study at IIT, K.

I have no words to express my thanks to my parents and my family members who have been constant source of inspiration to me. I wish to thank all my friends and well wishers who made my stay at IIT KANPUR, memorable and pleasant.

M.VENU GOPAL

Contents

ABSTRACT

This thesis presents the formulation of the structural dynamic equations of motion of a general rotor blade configuration with swept tip. The equations of motion have been derived using Hamilton's principle and Rayleigh Ritz finite element discretisation of the beam. First the formulation is validated by comparing the results of the present analysis for a straight blade with those available in the literature. Then a systematic structural dynamic analysis has been performed to identify the effect of tip sweep angles on the natural frequencies and mode shapes of the rotating blade.

Quantitative measures of lag-torsion and flap-torsion couplings have been defined based on the contribution of torsional deformation in the in-plane bending and out-of-plane bending modes. The effect of tip sweep on these couplings have been identified.

List of Figures

List of Tables

U	Strain energy
V_H	Hub velocity
$\vec{V}$	Velocity of a point 'p' on k^{th} blade
V_x, V_y, V_z	Components of $\vec{V}$ in X,Y and Z direction
$\{V^L\}, \{V^{NL}\}, \{V^I\}$	Vectors defined in the expression of kinetic energy variation
$\{V\}, \{W\}, \{U\}$	Vectors of element nodal degrees of freedom
v_k'	Spacial derivative$(=\frac{d(v_k/l)}{d(x/l)})$
w_k'	Spacial derivative$(=\frac{d(w_k/l)}{d(x/l)})$
W_e	External work due to nonconservative forces
x_k	Coordinate along k^{th} blade axis in n-th element
x, y, z	Coordinate of a point in the $\hat{e}_x - \hat{e}_y - \hat{e}_z$ system
x, η, ζ	Coordinate of a point in the $\hat{e}_x - \hat{e}_\eta - \hat{e}_\zeta$ system
$\bar{x}$	$=\frac{x}{l}$
$Z_u, Z_v, Z_w,$ $Z_\phi', Z_v', Z_w',$ $\bar{Z}_u, \bar{Z}_v, \bar{Z}_w,$ $\bar{Z}_\phi, \bar{Z}_v', \bar{Z}_w',$	Notation used for writing the beam kinetic energy in concise form
α	Warping amplitude
$\alpha_{xx}, \alpha_{x\eta}, \alpha_{x\zeta}$	Stress components
β_d	Blade predroop angle
β_p	Blade precone angle
β_s	Blade presweep angle
β_k	Local slope in flap bending of k^{th} blade
ξ_k	Local slope in lag bending of k^{th} blade
ϵ	Non dimensional parameter representing the order of magnitude of typical elastic blade bending slope
$\epsilon_{xx}, \epsilon_{x\eta}, \epsilon_{x\zeta}$	Strain components

τ_o	Initial twist rate of the blade
θ_G, θ_g	Geometric pitch in k^{th} blade
ϕ_k	Elastic twist
θ_I	Control pitch input
$\theta_x, \theta_y, \theta_z$	Rigid body perturbational rotation in roll-pitch-yaw
Λ_a	Tip anhedral angle (positive upward)
Λ_s	Tip sweep angle (positive backward sweep)
ρ	Density
$\{\Phi_c\}, \{\Phi_q\}$	Arrays of Hermite cubic and quardratic interpolation polynomials respectively
$\{\Phi_c'\}, \{\Phi_q'\}$	First derivative of $\{\Phi_c\}$ and $\{\Phi_q\}$ with respect to x
ψ_k	Azimuthal angle of k^{th} blade
ψ	Non-dimensional time ($\psi = \Omega t$)
Ψ	Cross-sectional warping function
$\vec{\omega}_k$	Angular velocity of k^{th} blade
Ω	Speed of rotation of rotor
$\omega_x, \omega_y, \omega_z$	Components of $\vec{\omega}_k$ in x,y and z direction
$(\ )'$	Differentiation of ( ) with respect to x
$d(\ )$	Differential of ( )
$(\ )_\eta, ()_\zeta$	Differentiation with respect to η and ζ
$(\ )_x, (\ )_{xx}$	Differentiation with respect to x of variables u,v ,w and ϕ
$\delta(\ )$	Variation of ( )
$\{\ \}$	Vector
$[\ ]$	Matrix
$1, 2, 3, 4, e, 5, 6$	Quantities refer to the corresponding coordinate system

Chapter 1

INTRODUCTION

It is well known that the time varying loads on the main rotor system contributes significantly to the vibration in helicopter. Therefore, the structural dynamic characteristics of the rotor blade and also the dynamic characteristics of the fuselage have a very strong influence on the vibratory levels in helicopter. Any analytical study of vibration in the helicopter requires the development of a dynamic model of the helicopter system. The major components of this model are:

- Rotor blade model
- Fuselage model
- Rotor-Fuselage interface model

The formulation of the rotor blade model requires the development of structural, inertial and aerodynamic operators associated with the rotor blade motion. The helicopter fuselage model is represented by an idealized structural dynamic model of a three-dimensional structure. The rotor-fuselage interface model must represent both the geometry of the interface as well as the aerodynamic interaction in an appropriate manner.

This study deals with the formulation and solution of a structural dynamic (structural

and inertia operators) model of a rotor blade with swept tip incorporating all complex geometric parameters of a realistic rotor system.

1.1 STRUCTURAL MODELLING OF ROTOR BLADE

Since helicopter rotor blades are long, slender beams undergoing moderate deformations, a nonlinear strain-displacement model is used to describe the coupling effects between axial, bending and torsion deformations. Generally, the strains are assumed to be small in comparison to unity. Such an assumption is consistent with the design requirement based on fatigue life consideration which states that the rotor blades must be designed to have an operating strain level well below the elastic limit of the blade material.

The first structural dynamic model for a rotor blade undergoing flap-lag-torsional deformations was developed by Houbolt and Brooks[1]. This model neglects the nonlinear coupling effects between bending and torsion, which were shown by later researchers, to be important for the rotor blade analysis.

Several moderate deflection beam theories have been developed [Refs. 2-7] for helicopter rotor blade analysis. These beam models were later used to formulate the structural, inertial and aerodynamic operators for the aeroelastic stability analysis of rotor blades [8-11]. Structural modelling for composite rotor blade has also been developed recently by various investigators [11-20]. The earlier models presented in Refs. 2-7, were restricted to treatment of isotropic blades. In general, these models did not include the effects of cross-sectional transverse shear and warping. On the other hand, the composite blade models include these effects.

In the eighties, efforts have been made to design rotor blades with modified tips to improve the rotor performance and loads. Subsequently several studies (Refs. 21-25)

were undertaken to address the influence of tip sweep on the aeroelastic stability and performance of rotor blades. A schematic of a rotor blade with tip sweep is shown in Fig. 1. Due to sweep (angle Λ_s defined in the plane of rotation) and anhedral (angle Λ_a defined in out-of-plane of rotation), rotor blades experience structural bending-torsion coupling effects. In addition due to high dynamic pressure, swept tips greatly influence the aeroelastic stability and response characteristics of the blade. Thus, tip sweep and anhedral angles can be effectively used to tailor the aeroelastic characteristics of the blades, there by imporve the noise and vibratory loads of the rotor.
During the derivation of the equation of motion of the rotor blade with swept tips, the formulation of inertia and aerodynamic operators generate a large number of higher order nonlinear terms. A consistent ordering scheme is used by assigning orders of magnitude to all the nondimensional parameters of the problem, in terms of blade bending slope (which is assumed to be of the order of ϵ). A second order approximation assumes that the terms of order ϵ^2 are neglected in comparison to terms of order 1. In this study a third order approximation has been used, i.e., terms of order ϵ^3 are neglected compared to terms of order 1. i.e.,

$$1 + O(\epsilon^3) \approx 1$$

In all the studies reported in Refs. 2-25, only certain geometric parameters such as precone, hinge offset and swept tips have been included in modelling the rotor blade dynamics. In the present formulation, an attempt has been made to include all the complex geometic parameters such as torque offset, blade root offset, precone angle, predroop angle, presweep angle, pretwist, tip sweep angle and tip anhedral angle, as well as hub motion.
This study is an extension of Ref. 26, which considered all geometric complexities of the rotor blade and hub motion, but did not consider tip sweep.

1.2 OBJECTIVES

The objectives of the present study are:

- To develop a most general beam type finite element model for the rotor blade with swept tips including various geometric parameters like root offset, torque offset, precone, predroop, presweep, pretwist and hub motion.

- To validate the model by comparing the results of this study with those available in literature.

- Conduct detailed studies on the dynamic behavior of rotor blade with straight and swept tips to determine the effects of bending-torsion coupling due to sweep and anhedral.

Chapter 2

ROTOR BLADE MODEL AND ASSUMPTIONS

Helicopter rotor blades are long slender beams attached to the hub through a complex geometrical and mechanical arrangement. The geometrical parameters describing the configuration of the rotor blade-hub system is shown in Fig. 1. The parameter 'a' represents the torque offset which is the distance from the centre of rotation (hub center) to the reference axis of the blade. 'e_1' and 'e_2' refer to the blade root offset distance from the hub. β_p represents the precone angle defining the orientation of the blade pitch control axis with respect to the hub plane. β_s and β_d represent the presweep and predroop angles respectively, representing the inclination of the blade reference axis with respect to the pitch control axis. The blade consists of a straight portion and a swept tip whose orientation relative to the straight portion is described by a sweep angle Λ_s and an anhedral angle Λ_a.

2.1 BASIC ASSUMPTIONS

In the forumulation of the dynamic model of the rotor blade with swept tip, several assumptions have been made which are given below.

1. The blade is treated as an elastic beam.

2. The blade is modeled by beam type finite elements along the elastic axis of the blade.

3. The speed of rotation (Ω) of rotor is constant.

4. The rotor shaft is rigid.

5. The blade undergoes moderate deformation in flap, lag, torsion and axial modes.

6. The blade cross section can have a general shape with distinct shear centre, aerodynamic centre and centre of mass.

7. The blade is assumed to be made of isotropic material but can have nonuniform properties along the span.

2.2 ORDERING SCHEME

In the formulation of the equations of motion of a rotor blade with swept tip undergoing moderate deformations, a large number of higher order terms are generated. In order to identify and eliminate higher order terms in a consistent manner, an ordering scheme is employed. This ordering scheme is based on the assumption that the slopes of the deformed elastic blade are moderate, and of order ϵ ($0.10 \leq \epsilon \leq 0.20$). Orders of magnitude are then assigned to various non-dimensional physical parameters governing the rotor blade dynamic problem, in terms of ϵ. In the derivation of the governing equations, it is assumed that terms of order ϵ are neglected with respect to terms of order 1, i.e.,

$$O(1) + O(\epsilon^3) \approx O(1)$$

The order of magnitude of various non-dimensional parameters governing this problem are given below:

<u>*ORDER*</u> 1

$cos\phi_k, sin\phi_k, \Lambda_a, \Lambda_s = O(1)$

$\frac{x_k}{l} = O(1)$

$\frac{1}{\Omega}\frac{\partial}{\partial t}() = \frac{\partial}{\partial \phi}() = O(1)$

$l\frac{\partial}{\partial x_k}() = \frac{\partial}{\partial x_k}() = O(1)$

$\underline{ORDER}\ \epsilon^{\frac{1}{2}}$

$\theta_{GK} = O(\epsilon^{\frac{1}{2}})$

$\underline{ORDER}\ \epsilon$

$\frac{a}{l}, \frac{e_1}{l}, \frac{e_2}{l}, \frac{\eta}{\zeta}, \frac{v_k}{l}, \frac{w_k}{l} = O(\epsilon)$

$v_k', w_k', \phi, \beta_p, \beta_d, \beta_s = O(\epsilon)$

$\underline{ORDER}\ \epsilon^{\frac{3}{2}}$

$Im_{\eta\eta}, Im_{\zeta\zeta} = O(\epsilon^{\frac{3}{2}})$

$\frac{R_x}{l}, \frac{R_y}{l}, \frac{R_z}{l}, \theta_x, \theta_y, \theta_z = O(\epsilon^{\frac{3}{2}})$

$\underline{ORDER}\ \epsilon^2$

$\frac{u_k}{l}, u_k', m\eta_m, m\zeta_m = O(\epsilon^2)$

It is important to note that ordering schemes are based on physical understanding of the behaviour of actual blade configurations. Hence care must be excercised in deleting higher order terms, based on this ordering scheme.

Chapter 3

COORDINATE SYSTEMS

The description of the complex deformation of a rotor blade requires several coordinate systems. The transformation relation between quantities referred in various inertial, non-inertial coordinate systems is to be established before deriving the equations of motion of the rotor blade. The relation between two orthogonal systems X_i, Y_i, Z_i and X_j, Y_j, Z_j with $\hat{e_{xi}}, \hat{e_{yi}}, \hat{e_{zi}}$ and $\hat{e_{xj}}, \hat{e_{yj}}, \hat{e_{zj}}$ as unit vectors along the respective axes can be written as

$$\left\{ \begin{array}{c} \hat{e}_{xi} \\ \hat{e}_{yi} \\ \hat{e}_{zi} \end{array} \right\} = [\mathbf{T_{ij}}] \left\{ \begin{array}{c} \hat{e}_{xj} \\ \hat{e}_{yj} \\ \hat{e}_{zj} \end{array} \right\} \tag{3.1}$$

where the transformation matrix $[\mathbf{T_{ij}}]$ can be obtained using the Euler angles required to rotate the j-system so as to make it parallel to i-system. The coordinate systems used in deriving the equation of motion for the rotor model are described below:

1 HUB FIXED INERTIAL SYSTEM-R

The coordinate system 'R', shown in Fig. 2, has its origin at the centre O_H of the unperturbed hub. The x_R axis is pointing towards the helicopter tail and Z_R is pointing upwards. The unit vectors along the three axes are $\hat{e}_{xR}, \hat{e}_{yR}, \hat{e}_{zR}$.

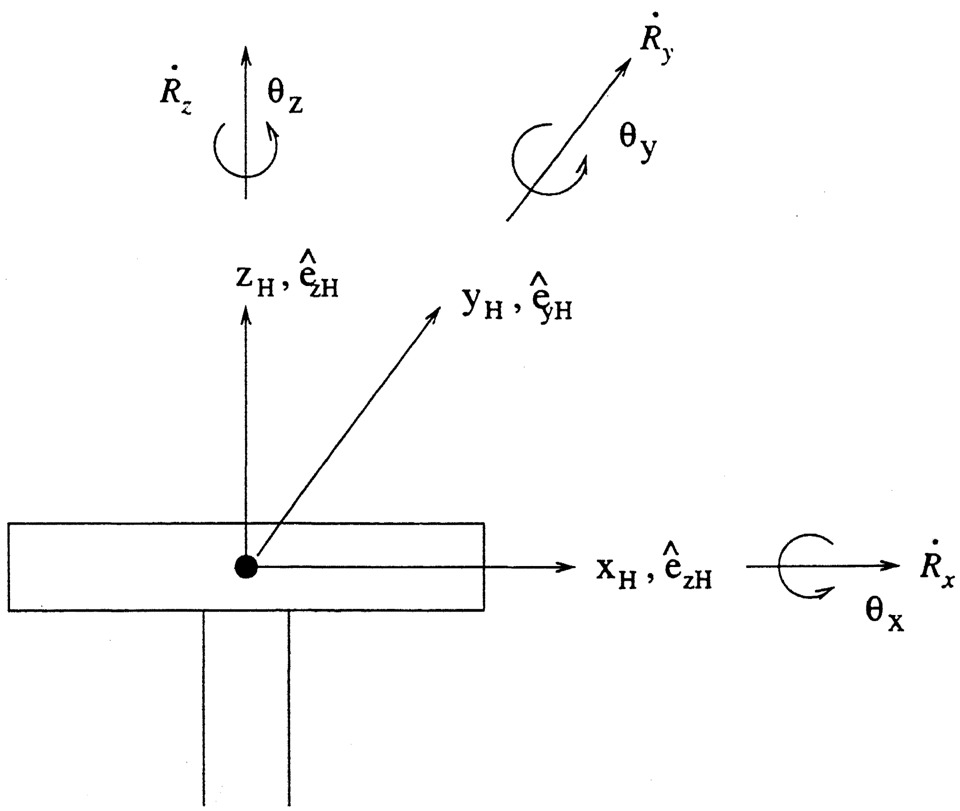


Figure 3: Body Fixed Hub System -H

3.2 HUB FIXED MOVING SYSTEM-H

The coordinate system H shown in Fig. 3, is a body fixed system with its origin fixed at the centre of rotor hub O_H. Prior to perturbational motion of the hub, H-system coincides with R-system. If θ_x, θ_y and θ_z represent the sequential yaw-pitch-roll rotations, then the transformation matrix $[\mathbf{T_{HR}}]$ can be written as-

$$[\mathbf{T_{HR}}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & \sin\theta_x \\ 0 & -\sin\theta_x & \cos\theta_x \end{bmatrix} \begin{bmatrix} \cos\theta_y & 0 & -\sin\theta_y \\ 0 & 1 & 0 \\ \sin\theta_y & 0 & \cos\theta_y \end{bmatrix} \begin{bmatrix} \cos\theta_z & \sin\theta_z & 0 \\ -\sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (3.2)$$

Since θ_x, θ_y and θ_z are assumed to be of order $\epsilon^{\frac{3}{2}}$, sine and cosine functions can be approximated as

$$\sin\theta \approx \theta, \quad \cos\theta \approx 1$$

Substituting the approximation, $[\mathbf{T_{HR}}]$ can be simplified as

$$[\mathbf{T_{HR}}] = \begin{bmatrix} 1 & \theta_z & -\theta_y \\ \theta_y\theta_z - \theta_z & 1 & \theta_x \\ \theta_x\theta_z + \theta_y & \theta_y\theta_z - \theta_x & 1 \end{bmatrix} \quad (3.3)$$

3.3 HUB FIXED ROTATING SYSTEM-1

The coordinate system 1 shown in Fig. 4, rotates about z_{1H} axis with constant angular speed Ω of the rotor. Its origin is fixed at hub centre O_H. This system can be obtained by rotating H system by an azimuthal angle ψ_k of the k^{th} blade about z_{1H} axis. The transformation matrix is given as -

$$[\mathbf{T_{1H}}] = \begin{bmatrix} \cos\psi_k & \sin\psi_k & 0 \\ -\sin\psi_k & \cos\psi_k & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (3.4)$$

where, ψ_k is azimuth angle of k^{th} blade

$$\psi_k = \psi + (k-1)\frac{2\pi}{N_b} \quad and \quad \psi = \Omega t$$

3.4 ROTATING SYSTEM-2K

The coordinate system 2K shown in Fig. 5 is a blade fixed coordinate system which rotates with k^{th} blade. The origin of the 2K system is at the location of the blade root A (Fig. 1), which is at a distance $a\hat{e}_{y1} + e_1\hat{e}_{x1}$ from the hub centre. 1-system and 2K-system are parallel.

3.5 PRECONED, ROTATING SYSTEM-3K

The system 3K shown in Fig. 6a and Fig. 6b also rotates with blade. This system is obtained by rotating 2K-sytem by an angle-β_p(precone angle) about y_{2k} axis. The transformation matrix between 2K and 3K system is given as-

$$[\mathbf{T_{32}}] = \begin{bmatrix} 1 & 0 & \beta_p \\ 0 & 1 & 0 \\ -\beta_p & 0 & 1 \end{bmatrix} \tag{3.5}$$

3.6 PREDROOPED, PRESWEPT, PITCHED, BLADE-FIXED ROTATING SYSTEM-4K

The 4K system shown in Fig. 6a and Fig. 6b, is blade fixed system with its origin at pitch bearing B of the blade.

It may be noted that the pitch axis of the blade is along $\hat{e}_{x3k}$ direction and the blade reference elastic axis is along the $\hat{e}_{x4k}$ direction. While changing the control pitch input of the blade, the elastic axis will move on the surface of a cone whose vertex angle is described by the angles β_s and β_d as shown in Fig. 6c.

The 4k system is obtained by the following steps-

Step-1 Translating the origin of 3K system by a distance 'e_2' along $\hat{e}_{x3k}$

Step-2 Then rotating the system by an angle -β_s(preweep angle) about z_{3k} axis.

Step-3 Then rotating the system by an angle -β_d(predroop angle) about y_{3k}.

Step-4 Then rotating the system by an angle θ_I(pitch input) about x_{3k}.

The transformation matrix is given as-

$$[\mathbf{T_{43}}] = \begin{bmatrix} 1 & -(\beta_s\cos\theta_I + \beta_d\sin\theta_I) & (\beta_d\cos\theta_I - \beta_s\sin\theta_I) \\ (\beta_s\cos\theta_I + \beta_d\sin\theta_I) & \cos\theta_I & \sin\theta_I \\ (-\beta_d\cos\theta_I + \beta_s\sin\theta_I) & -\sin\theta_I & \cos\theta_I \end{bmatrix} \tag{3.6}$$

3.7 UNDEFORMED ELEMENT COORDINATE SYSTEM-e

The e system, shown in Fig. 7, has its origin at the inboard node of the finite element. The vector $\hat{e}_{xe}$, is aligned with the beam element elastic axis; while the vectors $\hat{e}_{ye}$ and $\hat{e}_{ze}$ are cross sectional coordinate axes. For the straight portion of the blade , the $(\hat{e}_{xe}, \hat{e}_{ye}, \hat{e}_{ze})$ system has the same orientation as $(\hat{e}_{x4k}, \hat{e}_{y4k}, \hat{e}_{z4k})$ system. For the swept-tip element, the e system is oriented by rotating the 4k system about $\hat{e}_{z4k}$ by sweep angle Λ_s, and then about $\hat{e}_{y4k}$ by the anhedral angle Λ_a.

The transformation matrix between 4k and e system is given as-

$$\begin{Bmatrix} \hat{e}_{xe} \\ \hat{e}_{ye} \\ \hat{e}_{ze} \end{Bmatrix} = [\mathbf{T_{e4}}] \begin{Bmatrix} \hat{e}_{x4k} \\ \hat{e}_{y4k} \\ \hat{e}_{z4k} \end{Bmatrix} \tag{3.7}$$

For the element in the straight portion of the blade

$$[\mathbf{T_{e4}}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.8}$$

For the swept-tip element

$$[\mathbf{T_{e4}}] = \begin{bmatrix} \cos\Lambda_s \cos\Lambda_a & -\sin\Lambda_s & \cos\Lambda_s \sin\Lambda_a \\ \sin\Lambda_s \cos\Lambda_a & \cos\Lambda_s & \sin\Lambda_s \sin\Lambda_a \\ -\sin\Lambda_a & 0 & \cos\Lambda_a \end{bmatrix} \tag{3.9}$$

where, Λ_s is the blade tip sweep angle, positive for backward sweep, and Λ_a is the blade tip anhedral angle, positive upward.

3.8 ROTATING, BLADE-FIXED SYTEM-5K

The 5k system, shown in Fig. 8 is a cross-sectional coordinate system of the k^{th} blade. In the undeformed state of the blade, both e and 5k systems are parallel. But the origin of the 5k system is at a distance x_k from the origin of the e system. During elastic deformation of the blade, the 5k system translates and rotates with the cross-section. After deformation, the origin of the 5k system, from the origin of 4K system, is at the location given by

$$(\sum_{i=1}^{n-1}(l_e)_i)\hat{e}_{x4k} + (x_k + u_k)\hat{e}_{xe} + v_k\hat{e}_{ye} + w_k\hat{e}_{ze} \tag{3.10}$$

The transformation matrix between e and 5k sytem is obtained following a flap -lag sequence of rotation. The Euler angles are respectively -β_k and ξ_k corresponding to the local slope of the deformed blade in flap and lag directions. The transformation matrix is given by:

$$[\mathbf{T_{5e}}] = \begin{bmatrix} \cos\xi_k & \sin\xi_k & 0 \\ -\sin\xi_k & \cos\xi_k & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\beta_k & 0 & \sin\beta_k \\ 0 & 1 & 0 \\ -\sin\beta_k & 0 & \cos\beta_k \end{bmatrix} \tag{3.11}$$

Since the angle ξ_k and β_k are of the order O(ϵ), the transformation matrix can be simplified by noting that-

$$\sin\theta \approx \theta \;\; cos\theta \approx 1$$

The Euler angles can be expressed in the terms of the local slope of elastic deformation of the blade as

$$\beta_k = w_k' \approx \frac{dw_k}{dx}$$

$$\xi_k = v_k' \approx \frac{dv_k}{dx}$$

$$Note : x = \sum_{i=1}^{n-1} (l_e)_i + x_k$$

Substituting the above relations in the transformation matrix $[\mathbf{T_{5e}}]$ yields

$$[\mathbf{T_{5e}}] = \begin{bmatrix} 1 & v_k' & w_k' \\ -v_k' & 1 & -v_k' w_k' \\ -w_k' & 0 & 1 \end{bmatrix} \tag{3.12}$$

3.9 COORDINATE SYSTEM-6K

The 6K system shown in Fig. 9 represents the cross-sectional coordinate system in the deformed configuration of the blade. $\hat{e}_\eta - \hat{e}_\zeta$ represent the directions of the cross-sectional principal axes. 6K-system is obtained by rotating 5K system about $\hat{e}_{x5k}$ through the angle $(\phi_k + \theta_G)$, where θ_G represents the geometric twist angle of the cross-section and ϕ_k represents the elastic twist.

The transformation relation is given as-

$$\begin{Bmatrix} \hat{e}_x \\ \hat{e}_y \\ \hat{e}_z \end{Bmatrix}_{5k} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_G + \phi_k) & -\sin(\theta_G + \phi_k) \\ 0 & \sin(\theta_G + \phi_k) & \cos(\theta_G + \phi_k) \end{bmatrix} \begin{Bmatrix} \hat{e}_x \\ \hat{e}_\eta \\ \hat{e}_\zeta \end{Bmatrix}_{6k} \tag{3.13}$$

Chapter 4

KINEMATICS

During operation, the rotor blade undergoes deformation in in-plane bending (lag), out-of-plane bending (flap), torsion and axial modes. In addition, the hub centre has both translational (R_x, R_y, R_z) and rotational $(\theta_x, \theta_y, \theta_z)$ perturbational motion. The formulation of inertia operator and aerodynamic operator requires a proper discription of kinematics of the blade motion. In this section, an expression for the absolute velocity vector at any arbitrary point 'p' on the blade is derived.

4.1 POSITION VECTOR OF A POINT

The position vector of any arbitrary point 'p' in the n^{th} finite element of the deformed blade with respect to the hub centre O_H, is given by

$$\begin{aligned} \vec{r_p} &= a\hat{e}_{y1} + e_1\hat{e}_{x2k} + e_2\hat{e}_{x3k} + \sum_{i=1}^{n-1}(l_e)_i\hat{e}_{x4k} + (x_k + u_k)\hat{e}_{xe} \\ &\quad + v_k\hat{e}_{ye} + w_k\hat{e}_{ze} + \eta\hat{e}_\eta + \zeta\hat{e}_\zeta \end{aligned} \tag{4.1}$$

Transforming all the unit vectors to the 4K-system and neglecting the higher order terms, the position vector can be written as:

$$
\begin{aligned}
\vec{r}_p \;=\; & \hat{e}_{x4k} l[-a(\beta_s \cos\theta_I + \beta_d \sin\theta_I) + e_1 + e_2 + \sum_{i=1}^{n-1}(l_e)_i \\
& + \cos\Lambda_s \cos\Lambda_a (x_k + u_k) + v_k \sin\Lambda_s \cos\Lambda_a - w_k \sin\Lambda_a \\
& + \eta\{-v_k' \cos\Lambda_s \cos\Lambda_a \cos(\theta_G + \phi_k) - w_k' \cos\Lambda_s \cos\Lambda_a \sin(\theta_G + \phi_k) \\
& - \sin\Lambda_s \cos(\theta_G + \phi_k) + \cos\Lambda_s \sin\Lambda_a \sin(\theta_G + \phi_k)\} \\
& + \zeta\{v_k' \cos\Lambda_s \cos\Lambda_a \sin(\theta_G + \phi_k) - w_k' \cos\Lambda_s \cos\Lambda_a \cos(\theta_G + \phi_k) \\
& + \sin\Lambda_s \sin(\theta_G + \phi_k) + \cos\Lambda_s \sin\Lambda_a \cos(\theta_G + \phi_k)\}] \\
& + \hat{e}_{y4k} l[a \cos\theta_I + e_1(\beta_s \cos\Lambda_I + \beta_d \sin\theta_I - \beta_p \sin\theta_I) + e_2(\beta_s \cos\theta_I \\
& + \beta_d \sin\theta_I) \sin\Lambda_s (x_k + u_k) + v_k \cos\Lambda_s + \eta\{-v_k' \sin\Lambda_s \cos\Lambda_a \cos(\theta_G + \phi_k) \\
& - w_k' \sin\Lambda_s \cos\Lambda_a \sin(\theta_G + \phi_k) + \cos\Lambda_s \cos(\theta_G + \phi_k) \\
& + \sin\Lambda_s \sin\Lambda_a \sin(\theta_G + \phi_k)\} + \zeta\{v_k' \sin\Lambda_s \cos\Lambda_a \sin(\theta_G + \phi_k) \\
& - w_k' \sin\Lambda_s \cos\Lambda_a \sin(\theta_G + \phi_k) - \cos\Lambda_s \sin(\theta_G + \phi_k) + \sin\Lambda_s \sin\Lambda_a \cos(\theta_G + \phi_k)\}] \\
& + \hat{e}_{z4k} l[-a \sin\theta_I + e_1(-\beta_d \cos\theta_I + \beta_s \sin\theta_I - \beta_p \cos\theta_I) + e_2(-\beta_d \cos\theta_I \\
& + \beta_s \sin\theta_I) + \cos\Lambda_s \sin\Lambda_a (x_k + u_k) + v_k \sin\Lambda_s \sin\Lambda_a + w_k \cos\Lambda_a \\
& + \eta\{v_k' \sin\Lambda_a \cos(\theta_G + \phi_k) + w_k' \sin\Lambda_a \sin(\theta_G + \phi_k) \\
& + \cos\Lambda_a \sin(\theta_G + \phi_k)\} + \zeta\{w_k' \sin\Lambda_a \cos(\theta_G + \phi_k) - v_k' \sin\Lambda_a \sin(\theta_G + \phi_k) \\
& + \cos\lambda_a \cos(\theta_G + \phi_k)\}]
\end{aligned}
\tag{4.2}
$$

In Eq. 4.2 all the length quantities are nondimensionalized with respect to the length of the blade $'l'$. For the sake of convenience, the nondimensional length quantities are referred without an overbar$(\bar{})$. The expression given in Eq. 4.2 is general, in the sense that for position vector of any point in the straight portion of blade Λ_a and Λ_s are set equal to zero. Eq. 4.2 can also be written in symbolic form as

$$\vec{r}_p = l[r_x\hat{e}_{x4k} + r_y\hat{e}_{y4k} + r_z\hat{e}_{z4k}]$$

4.2 ANGULAR VELOCITY VECTOR

The angular velocity vector $\vec{\omega}$ of k^{th} blade consists of three components. They are

- The constant rotational speed (Ω) of the rotor;
- the rigid body angular velocity ($\vec{\omega}_{rigid}$) of the hub due to perturbational rotation in roll-pitch-yaw (θ_x, θ_y and θ_z)
- the angular velocity contribution due to the rate of change of control pitch input $\Omega\dot{\theta}_I$ to the blade

$$\vec{\omega} = \Omega\hat{e}_{zH} + \vec{\omega}_{rigid} + \Omega\dot{\theta}_I\hat{e}_{x3k} \tag{4.3}$$

where

$$\vec{\omega}_{rigid} = \Omega\dot{\theta}_x\hat{e}_{x1k} + \Omega\dot{\theta}_y\hat{e}_{y1k} + \Omega\dot{\theta}_z\hat{e}_{z1k} \tag{4.4}$$

where $\dot{(\ )}$ indicates differentiation with respect to the nondimensional time ψ,

$$\psi = \Omega t$$

Note that

$$\frac{d}{dt}(.) = \Omega\frac{d}{d\Omega t}(\ ) = \Omega\frac{d}{d\psi}(\ ) = \Omega(\dot{\ })$$

Transforming all the unit vectors of Eq. 4.3 to the 4K-system and neglecting the higher order terms, the angular velocity of k^{th} blade can be written in symbolic form as:

$$\vec{\omega} = \Omega[\omega_x\hat{e}_{x4k} + \omega_y\hat{e}_{y4k} + \omega_z\hat{e}_{z4k}] \tag{4.5}$$

where

$$\begin{aligned}
\omega_x &= [\dot{\theta}_x \cos\psi_k + \dot{\theta}_y \sin\psi_k + \dot{\theta}_I + (\beta_p + \beta_d \cos\theta_I \\
&\quad -\beta_s \sin\theta_I)] && (4.6) \\
\omega_y &= [-\dot{\theta}_x \cos\theta_I \sin\psi_k + \dot{\theta}_y \cos\theta_I \cos\psi_k + \dot{\theta}_z \sin\theta_I \\
&\quad +\dot{\theta}_I(\beta_s \cos\theta_I + \beta_d \sin\theta_I)] \\
\omega_z &= [\dot{\theta}_x \sin\theta_I \sin\psi_k - \dot{\theta}_y \sin\theta_I \cos\psi_k + \dot{\theta}_z \cos\theta_I \\
&\quad + \cos\theta_I + \dot{\theta}_I(-\beta_d \cos\theta_I + \beta_s \sin\theta_I)] && (4.7) \\
& && (4.8)
\end{aligned}$$

4.3 VELOCITY AT POINT 'P'

The absolute velocity vector $\vec{V}$, at point 'p' on the deformed beam can be written as:

$$\vec{V} = \vec{V}_H + \dot{\vec{r}}_p + (\vec{\omega} \times \vec{r_p}) \qquad (4.9)$$

where $\vec{V}_H$ is the rigid body perturbational translation of the hub center O_H which is given as:

$$\vec{V}_H = \Omega l[\dot{R}_x \hat{e}_{xR} + \dot{R}_y \hat{e}_{yR} + \dot{R}_z \hat{e}_{zR}] \qquad (4.10)$$

Transforming all the unit vectors to 4K-system

$$\vec{V}_H = (V_H)_x \hat{e}_{x4k} + (V_H)_y \hat{e}_{y4k} + (V_H)_z \hat{e}_{z4k} \qquad (4.11)$$

where

$$\begin{aligned}
(V_H)_x &= \Omega l \dot{R}_x \{\cos\psi_k + \sin\psi_k(\beta_s \cos\theta_I + \beta_d \sin\theta_I)\} + \\
&\quad \Omega l \dot{R}_y \{\cos\psi_k(-\beta_s \cos\theta_I - \beta_d \sin\theta_I) + \sin\psi_k\} + \\
&\quad \Omega l \dot{R}_z \{\beta_p + \beta_d \cos\theta_I - \beta_s \sin\theta_I\} && (4.12) \\
(V_H)_y &= \Omega l \dot{R}_x \{\cos\psi_k(\beta_s \cos\theta_I + \beta_d \sin\theta_I - \beta_p \sin\theta_I) - \sin\psi_k \cos\theta_I\} +
\end{aligned}$$

$$
\begin{aligned}
& \Omega l\dot{R}_y\{\cos\psi_k\cos\theta_I+\sin\psi_k(\beta_s\cos\theta_I+\beta_d\sin\theta_I-\beta_p\sin\theta_I\}+ \\
& \Omega l\dot{R}_z\{\sin\theta_I\} \qquad (4.13) \\
(V_H)_z \;=\; & \Omega l\dot{R}_x\{\cos\psi_k(-\beta_d\cos\theta_I+\beta_s\sin\theta_I-\beta_p\cos\theta_I)+\sin\psi_k\sin\theta_I\}+ \\
& \Omega l\dot{R}_y\{\cos\psi_k(-\sin\theta_I)+\sin\psi_k(-\beta_d\cos\theta_I+\beta_s\sin\theta_I-\beta_p\cos\theta_I\}+ \\
& \Omega l\dot{R}_z\{\cos\theta_I\}
\end{aligned}
$$

$$(4.14)$$

and

$$
\begin{aligned}
\dot{\vec{r}}_p \;=\; & \hat{e}_{x4k}l\Omega[-a\dot{\theta}_I\;(-\beta_s\;\sin\theta_I\;+\beta_d\;\cos\theta_I\;)+\cos\Lambda_s\;\cos\Lambda_a\;\dot{u}_k] \\
& +\dot{v}_k\sin\Lambda_s\;cos\Lambda_a\;-\dot{w}_k\sin\Lambda_a\;+\eta\{-\dot{v}_k'\;\cos\Lambda_s\;\cos\Lambda_a\;\cos(\theta_G+\phi_k) \\
& +v_k'\;\dot{\phi}_k\;\cos\Lambda_s\;\cos\Lambda_a\;\sin(\theta_G+\phi_k)\;-\dot{w}_k'\;\cos\Lambda_s\;\cos\Lambda_a\;\sin(\theta_G+\phi_k) \\
& -w_k'\;\cos\Lambda_s\;\cos\Lambda_a\;\dot{\phi}_k\;\cos(\theta_G+\phi_k)\;+\dot{\phi}_k\;\sin\Lambda_s\;\sin(\theta_G+\phi_k) \\
& +\dot{\phi}_k\;\cos\Lambda_s\;\sin\Lambda_a\;\cos(\theta_G+\phi_k)\;\} \\
& +\zeta\{-\dot{w}_k'\;\cos\Lambda_s\;\cos\Lambda_a\;\cos(\theta_G+\phi_k)\;+w_k'\;\cos\Lambda_s\;\cos\Lambda_a\;\sin(\theta_G+\phi_k) \\
& +\dot{v}_k'\;\cos\Lambda_s\;\cos\Lambda_a\;\sin(\theta_G+\phi_k)\;+v_k'\;\cos\Lambda_s\;\cos\Lambda_a\;\dot{\phi}_k\;\cos(\theta_G+\phi_k) \\
& +\sin\Lambda_s\;\dot{\phi}_k\;\cos(\theta_G+\phi_k)\;-\dot{\phi}_k\;\cos\Lambda_s\;\sin\Lambda_a\;\sin(\theta_G+\phi_k)\;\}] \\
& +\hat{e}_{y4k}l\Omega[-a\dot{\theta}_I\;\sin\theta_I\;+e_1\dot{\theta}_I\;(-\beta_s\;\sin\theta_I\;+\beta_d\;\cos\theta_I\;-\beta_p\;\cos\theta_I\;) \\
& +e_2\dot{\theta}_I\;(-\beta_s\;\sin\theta_I\;+\beta_d\;\cos\theta_I\;)-\dot{u}_k\sin\Lambda_s\;+\dot{v}_k\cos\Lambda_s \\
& +\eta\{-\dot{v}_k'\;\sin\Lambda_s\;\cos\Lambda_a\;\cos(\theta_G+\phi_k)\;+v_k'\;\sin\Lambda_s\;\cos\Lambda_a\;\dot{\phi}_k\;\sin(\theta_G+\phi_k) \\
& -\dot{w}_k'\;\sin\Lambda_s\;\cos\Lambda_a\;\sin(\theta_G+\phi_k)\;-w_k'\;\dot{\phi}_k\;\sin\Lambda_s\;\cos\Lambda_a\;\cos(\theta_G+\phi_k) \\
& -\dot{\phi}_k\;\cos\Lambda_s\;\sin(\theta_G+\phi_k)\;+\dot{\phi}_k\;\sin\Lambda_s\;\sin\Lambda_a\;\cos(\theta_G+\phi_k)\;\} \\
& +\zeta\{-\dot{w}_k'\;\sin\Lambda_s\;\cos\Lambda_a\;\cos(\theta_G+\phi_k)\;+w_k'\;\dot{\phi}_k\;\sin\Lambda_s\;\cos\Lambda_a\;\sin(\theta_G+\phi_k) \\
& +\dot{v}_k'\;\sin\Lambda_s\;\cos\Lambda_a\;\sin(\theta_G+\phi_k)\;+v_k'\;\sin\Lambda_s\;\cos\Lambda_a\;\dot{\phi}_k\;\cos(\theta_G+\phi_k) \\
& -\dot{\phi}_k\;\cos\Lambda_s\;\sin(\theta_G+\phi_k)\;-\dot{\phi}_k\;\sin\Lambda_s\;\sin\Lambda_a\;\sin(\theta_G+\phi_k)\;\}] \\
& +\hat{e}_{z4k}l\Omega[-a\dot{\theta}_I\;\cos\theta_I\;+e_1\dot{\theta}_I\;(\beta_d\;\sin\theta_I\;+\beta_s\;\cos\theta_I\;+\beta_p\;\sin\theta_I\;)
\end{aligned}
$$

$$+e_2\dot{\theta}_I\ (\beta_d\ \sin\theta_I\ +\beta_s\ \cos\theta_I\ ) + \dot{u}_k \cos\Lambda_s\ \sin\Lambda_a$$
$$+\dot{v}_k \sin\Lambda_s\ \sin\Lambda_a\ +\dot{w}_k \cos\Lambda_a$$
$$+\eta\{\dot{v}_k'\ \sin\Lambda_a\ \cos(\theta_G+\phi_k)\ -v_k'\ \dot{\phi}_k\ \sin\Lambda_a\ \sin(\theta_G+\phi_k)\ +\dot{w}_k' \sin\Lambda_a\ \sin(\theta_G+\phi_k)$$
$$+w_k'\ \dot{\phi}_k\ \sin\Lambda_a\ \cos(\theta_G+\phi_k)\ +\dot{\phi}_k\ \cos\Lambda_a\ \cos(\theta_G+\phi_k)\ \}$$
$$+\zeta\{\dot{w}_k'\ \sin\Lambda_a\ \cos(\theta_G+\phi_k)\ -w_k'\ \sin\Lambda_a\ \dot{\phi}_k\ \sin(\theta_G+\phi_k)\ -\dot{v}_k'\ \sin\Lambda_a\ \sin(\theta_G+\phi_k)$$
$$-v_k'\ \sin\Lambda_a\ \dot{\phi}_k\ \cos(\theta_G+\phi_k)\ -\dot{\phi}_k\ \cos\Lambda_a\ \sin(\theta_G+\phi_k)\ \}]$$

(4.15)

and

$$(\vec{\omega}\times\vec{r}) = \begin{vmatrix} \hat{e}_{x4k} & \hat{e}_{y4k} & \hat{e}_{z4k} \\ \omega_x & \omega_y & \omega_z \\ r_x & r_y & r_z \end{vmatrix} \tag{4.16}$$

where r_x, r_y, r_z and $\omega_x, \omega_y, \omega_z$ are the x,y and z components of $\vec{r}_p$ and $\vec{\omega}$ respectively. Substituting various quantities in Eq. 4.9 from Eqs. 4.2 ,4.6-4.8 and 4.11-4.16, the velocity at point p is obtained. This can be written in symbolic form as:

$$\vec{V} = \Omega l[V_x\hat{e}_{x4k} + V_y\hat{e}_{y4k} + V_z\hat{e}_{z4k}] \tag{4.17}$$

The components of the velocity vector are given below.

$$\begin{aligned}
\vec{V}_x &= (V_H)_x + (\dot{r}_p)_x + (w_y r_z - w_z r_y) \\
&= \dot{R}_x\{cos\psi_k + sin\psi_k(\beta_s cos\theta_I + \beta_d sin\theta_I)\} \\
&+\dot{R}_y\{cos\psi_k(-\beta_s cos\theta_I - \beta_d sin\theta_I) + sin\psi_k\} + \\
&\dot{R}_z\{\beta_p + \beta_d cos\theta_I - \beta_s sin\theta_I\} \\
&+a\dot{\theta}_I(\beta_s sin\theta_I - \beta_d cos\theta_I) + \dot{u}_k cos\Lambda_s cos\Lambda_a + \dot{v}_k sin\Lambda_s cos\Lambda_a - \dot{w}_k sin\Lambda_a \\
&+\eta[cos\Lambda_s cos\Lambda_a\{-\dot{v}'_k cos(\theta_G + \phi_k) \\
&+v'_k\dot{\phi}_k sin(\theta_G + \phi_k - \dot{w}'_k sin(\theta_G + \phi_k) \\
&-w'_k\dot{\phi}_k cos(\theta_G + \phi_k)\} + \dot{\phi}_k sin(\theta_G + \phi_k) sin\Lambda_s \\
&+\dot{\phi}_k cos(\theta_G + \phi_k) cos\Lambda_s sin\Lambda_a] \\
&+\zeta[cos\Lambda_s cos\Lambda_a\{-\dot{w}'_k cos(\theta_G + \phi_k) \\
&+w'_k\dot{\phi}_k sin(\theta_G + \phi_k + \dot{v}'_k sin(\theta_G + \phi_k) \\
&+v'_k\dot{\phi}_k cos(\theta_G + \phi_k)\} + \dot{\phi}_k cos(\theta_G + \phi_k) sin\Lambda_s \\
&-\dot{\phi}_k sin(\theta_G + \phi_k) cos\Lambda_s sin\Lambda_a] + (\dot{\theta}_I(\beta_s cos\theta_I + \beta_d sin\theta_I) \\
&+cos\theta_I(-\dot{\theta}_x sin\psi_k + \dot{\theta}_y cos\psi_k) + \dot{\theta}_z sin\theta_I) \\
&[-a sin\theta_I + (e_1 + e_2)\{-\beta_d cos\theta_I + \beta_s sin\theta_I\} \\
&-e_1\beta_p cos\theta_I + (x_k + u_k) cos\Lambda_s sin\Lambda_a + v_k sin\Lambda_s sin\Lambda_a \\
&+w_k cos\Lambda_a + \eta[v'_k cos(\theta_G + \phi_k) sin\Lambda_a \\
&+w'_k sin(\theta_G + \phi_k) sin\Lambda_a + sin(\theta_G + \phi_k) cos\Lambda_a] \\
&+\zeta[w'_k cos(\theta_G + \phi_k) sin\Lambda_a - v'_k sin(\theta_G + \phi_k) sin\Lambda_a \\
&+cos(\theta_G + \phi_k) cos\Lambda_a]] - (\dot{theta}_I(-\beta_d cos\theta_I \\
&+\beta_s sin\theta_I) + cos\theta_I(1 + \dot{\theta}_z) + sin\theta_I(\dot{\theta}_x sin\psi_k - \dot{\theta} cos\psi_k)) \\
&[a cos\theta_I + (e_1 + e_2)\{\beta_s cos\theta_I + \beta_d sin\theta_I\} \\
&-e_1\beta_p sin\theta_I - (x_k + u_k) sin\Lambda_s + v_k cos\Lambda_s
\end{aligned}$$

$$
\begin{aligned}
&+\eta[sin\Lambda_s cos\Lambda_a\{-v_k' cos(\theta_G+\phi_k) - w_k' sin(\theta_G+\phi_k)\} \\
&+cos(\theta_G+\phi_k)cos\Lambda_s + sin(\theta_G+\phi_k sin\Lambda_s sin\Lambda_a] \\
&+\zeta[sin\Lambda_s cos\Lambda_a\{-w_k' cos(\theta_G+\phi_k) + v_k' sin(\theta_G+\phi_k)\} \\
&-sin(\theta_G+\phi_k)cos\Lambda_s + cos(\theta_G+\phi_k sin\Lambda_s sin\Lambda_a]]
\end{aligned} \tag{4.18}
$$

$$
\begin{aligned}
\vec{V}_y \;=\; & (V_H)_y + (\dot{r}_p)_y + (w_x r_z - w_z r_x) \\
& = \dot{R}_x\{\cos\psi_k\ (\beta_s\ \cos\theta_I\ + \beta_d\ \sin\theta_I\ - \beta_p\ \sin\theta_I\ ) - \sin\psi_k\ \cos\theta_I\ \} \\
& + \dot{R}_y\{\cos\psi_k\ \cos\theta_I\ + \sin\psi_k\ (\beta_s\ \cos\theta_I\ + \beta_d\ \sin\theta_I\ - \beta_p\ \sin\theta_I\ )\} \\
& + \dot{R}_z \sin\theta_I\ - a\dot{\theta}_I\ \sin\theta_I\ + \dot{\theta}_I\ \{(e_1 + e_2)(-\beta_s\ \sin\theta_I\ + \beta_d\ \cos\theta_I\ ) \\
& - \beta_p\ e_1 \cos\theta_I\ \} - \dot{u}_k \sin\Lambda_s\ + \dot{v}_k \cos\Lambda_s\ + \eta[\sin\Lambda_s\ \cos\Lambda_a \\
& \{-\dot{v}_k'\ \cos(\theta_G + \phi_k)\ + v_k'\ \dot{\phi}_k\ \sin(\theta_G + \phi_k)\ - \dot{w}_k'\ \sin(\theta_G + \phi_k) \\
& - w_k'\ \dot{\phi}_k\ \cos(\theta_G + \phi_k)\ \} - \dot{\phi}_k\ \cos\Lambda_s\ \sin(\theta_G + \phi_k)\ + \dot{\phi}_k\ \sin\Lambda_s\ \sin\Lambda_a\ \cos(\theta_G + \phi_k)\ ] \\
& + \zeta[\sin\Lambda_s\ \cos\Lambda_a\ \{-\dot{w}_k'\ \cos(\theta_G + \phi_k)\ + w_k'\ \dot{\phi}_k\ \sin(\theta_G + \phi_k) \\
& \dot{v}_k'\ \sin(\theta_G + \phi_k)\ + v_k'\ \dot{\phi}_k\ \cos(\theta_G + \phi_k)\ \} \\
& - \dot{\phi}_k\ \cos\Lambda_s\ \cos(\theta_G + \phi_k)\ - \dot{\phi}_k\ \sin\Lambda_s\ \sin\Lambda_a\ \sin(\theta_G + \phi_k)\ ] \\
& + \left(\dot{\theta}_I\ \{-\beta_d\ \cos\theta_I\ + \beta_s\ \sin\theta_I\ \} + (1 + \dot{\theta}_z\ ) \cos\theta_I\ + \sin\theta_I\ \{\dot{\theta}_x\ \sin\psi_k\ - \dot{\theta}_y\ \cos\psi_k\ \}\right) \\
& [-a(\beta_s\ \cos\theta_I\ + \beta_d\ \sin\theta_I\ ) + e_1 + e_2 + \sum_{e=1}^{n-1} l_e \\
& + (x_k + u_k) \cos\Lambda_s\ \cos\Lambda_a\ + v_k \sin\Lambda_s\ \cos\Lambda_a\ - w_k \sin\Lambda_a \\
& + \eta[\cos\Lambda_s\ \cos\Lambda_a\ \{-v_k'\ \cos(\theta_G + \phi_k)\ - w_k'\ \sin(\theta_G + \phi_k)\ \} \\
& - \cos(\theta_G + \phi_k)\ \sin\Lambda_s\ + \sin(\theta_G + \phi_k)\ \cos\Lambda_s\ \sin\Lambda_a\ ] \\
& + \zeta[\cos\Lambda_s\ \cos\Lambda_a\ \{-w_k'\ \cos(\theta_G + \phi_k)\ + v_k'\ \sin(\theta_G + \phi_k)\} \\
& + \sin\Lambda_s\ \sin(\theta_G + \phi_k)\ + \cos\Lambda_s\ \sin\Lambda_a\ \cos(\theta_G + \phi_k)\ ]] \\
& - \left(\dot{\theta}_I\ - \{\beta_p\ + \beta_d\ \cos\theta_I\ - \beta_s\ \sin\theta_I\ \} - \dot{\theta}_x\ \cos\psi_k\ - \dot{\theta}_y\ \sin\psi_k\ \right) \\
& [-a \sin\theta_I\ + (e_1 + e_2)\{-\beta_d\ \cos\theta_I\ + \beta_s\ \sin\theta_I\ \} - e_1\beta_p\ \sin\theta_I \\
& + (x_k + u_k) \cos\Lambda_s\ \sin\Lambda_a\ + v_k \sin\Lambda_s\ \sin\Lambda_a\ + w_k \cos\Lambda_a \\
& + \eta[v_k'\ \cos(\theta_G + \phi_k)\ \sin\Lambda_a\ + w_k'\ \sin(\theta_G + \phi_k)\ \sin\Lambda_a\ + \sin(\theta_G + \phi_k)\ \cos\Lambda_a\ ] \\
& + \zeta[-v_k'\ \sin(\theta_G + \phi_k)\ \sin\Lambda_a\ + w_k'\ \cos(\theta_G + \phi_k)\ \sin\Lambda_a\ + \cos(\theta_G + \phi_k)\ \cos\Lambda_a\ ]]
\end{aligned}
\tag{4.19}
$$

$$
\begin{aligned}
\vec{V}_z \;=\;& (V_H)_z + (\dot{r}_p)_z + (w_x r_y - w_y r_x) \\
&= \dot{R}_x\{\cos\psi_k \ (-\beta_d \ \cos\theta_I \ + \beta_s \ \sin\theta_I \ - \beta_p \ \cos\theta_I \ ) + \sin\psi_k \ \sin\theta_I \ \} \\
&+\dot{R}_y\{\cos\psi_k \ - \sin\theta_I \ + \sin\psi_k \ (-\beta_d \ \cos\theta_I \ + \beta_s \ \sin\theta_I \ - \beta_p \ \sin\theta_I \ )\} \\
&+\dot{R}_z \cos\theta_I \ - a\dot{\theta}_I \ \cos\theta_I \ + \dot{\theta}_I \ \{(e_1 + e_2)(\beta_d \ \sin\theta_I \ + \beta_s \ \cos\theta_I \ ) \\
&\beta_p \ e_1 \cos\theta_I \ \} + \dot{u}_k \cos\Lambda_s \ \sin\Lambda_a \ + \dot{v}_k \sin\Lambda_s \ \sin\Lambda_a \ + \dot{w}_k \cos\Lambda_a \ + \eta[\sin\Lambda_a \\
&\{\dot{v}'_k \ \cos(\theta_G + \phi_k) \ - v'_k \ \dot{\phi}_k \ \sin(\theta_G + \phi_k) \ - \dot{w}'_k \ \sin(\theta_G + \phi_k) \\
&+w'_k \ \dot{\phi}_k \ \cos(\theta_G + \phi_k) \ \} + \dot{\phi}_k \ \cos\Lambda_a \ \cos(\theta_G + \phi_k) \ ] \\
&+\zeta[\sin\Lambda_a \ \{\dot{w}'_k \ \cos(\theta_G + \phi_k) \ - w'_k \ \dot{\phi}_k \ \sin(\theta_G + \phi_k) \\
&-\dot{v}'_k \ \sin(\theta_G + \phi_k) \ + v'_k \ \dot{\phi}_k \ \cos(\theta_G + \phi_k) \ \} - \dot{\phi}_k \ \cos\Lambda_a \ \sin(\theta_G + \phi_k) \ ] \\
&+ \Big(\dot{\theta}_I \ + \{\beta_p \ + \beta_d \ \cos\theta_I \ - \beta_s \ \sin\theta_I \ \} + \dot{\theta}_x \ \cos\psi_k \ + \dot{\theta}_y \ \sin\psi_k \ \Big) \\
&[a\cos\theta_I \ + (\beta_s \ \cos\theta_I \ + \beta_d \ \sin\theta_I \ )(e_1 + e_2) - e_1\beta_p \ \sin\theta_I \ - (x_k + u_k)\sin\Lambda_s \\
&+v_k \cos\Lambda_s \ + \eta[\sin\Lambda_s \ \cos\Lambda_a \ \{-v'_k \ \cos(\theta_G + \phi_k) \ - w'_k \ \sin(\theta_G + \phi_k) \ \} \\
&+\cos(\theta_G + \phi_k) \ \cos\Lambda_s \ + \sin(\theta_G + \phi_k) \ \sin\Lambda_s \ \sin\Lambda_a \ ] \\
&+\zeta[\sin\Lambda_s \ \cos\Lambda_a \ \{-w'_k \ \cos(\theta_G + \phi_k) \ + v'_k \ \sin(\theta_G + \phi_k)\} \\
&-\sin\Lambda_s \ \cos(\theta_G + \phi_k) \ + \sin\Lambda_s \ \sin\Lambda_a \ \cos(\theta_G + \phi_k) \ ]] \\
&- \Big(\dot{\theta}_I \ \{\beta_s \ \cos\theta_I \ + \beta_d \ \sin\theta_I \ \} - (\dot{\theta}_x \ \sin\psi_k \ - \dot{\theta}_y \ \cos\psi_k \ )\cos\theta_I \ + \dot{\theta}_z \ \sin\theta_I \ \Big) \\
&[-a\{\beta_s \ \cos\theta_I \ + \beta_d \ \sin\theta_I \ \} + (e_1 + e_2) + \sum_{e=1}^{n-1} l_e \\
&+(x_k + u_k)\cos\Lambda_s \ \cos\Lambda_a \ + v_k \sin\Lambda_s \ \cos\Lambda_a \ - w_k \sin\Lambda_a \\
&+\eta[\cos\Lambda_s \ \cos\Lambda_a \ \{-v'_k \ \cos(\theta_G + \phi_k) \ - w'_k \ \sin(\theta_G + \phi_k) \ \} \\
&-\cos(\theta_G + \phi_k) \ \sin\Lambda_s \ + \cos\Lambda_s \ \sin\Lambda_a \ \sin(\theta_G + \phi_k) \ ] \\
&+\zeta[\cos\Lambda_s \ \cos\Lambda_a \ \{+v'_k \ \sin(\theta_G + \phi_k) \ - w'_k \ \cos(\theta_G + \phi_k) \ \} \\
&+\sin(\theta_G + \phi_k) \ \sin\Lambda_s \ + \cos\Lambda_s \ \sin\Lambda_a \ \cos(\theta_G + \phi_k) \ ]]
\end{aligned}
\tag{4.20}
$$

Chapter 5

EQUATIONS OF MOTION FOR ROTOR BLADE

The coupled equations of motion of the rotor blade can be derived using Hamilton's principle. The mathematical form of Hamilton's principle is stated as follows:

$$\int_{t_1}^{t_2} (\delta U - \delta T - \delta W_e)dt = 0 \tag{5.1}$$

where U is the strain energy; T is kinetic energy; W_e is the work done by the non-conservative external loads.

In this chapter, the expressions for the variation of kinetic energy and strain energy of the rotor blade are derived.

5.1 KINETIC ENERGY OF BLADE

The total kinetic energy of the beam, T, is defined as :

$$\begin{aligned} T &= \frac{1}{2}\sum_i \int\int\int_V \rho\vec{V}.\vec{V} dVol \\ &= \frac{1}{2}\sum_i \int_0^{(l_e)_i} \int\int_A \rho\vec{V}.\vec{V}\, d\eta\, d\zeta\, dx \end{aligned} \tag{5.2}$$

where $\vec{V}$ is the velocity of an arbitrary point 'p' on the blade cross section with respect to the inertial reference frame. The variation of kinetic energy, δT can be written as:

$$\delta T = \sum_i \int_0^{(l_e)_i} \int\int_A \rho \vec{V}.\delta\vec{V} d\eta\, d\zeta\, dx \tag{5.3}$$

Substituting for the velocity $\vec{V}$ from Eq. 4.17 and integrating δT by parts with respect to time, yields

$$\delta T = \sum_i \int_0^{(l_e)_i} \int\int_A \rho[Z_u\delta u + Z_v\delta v + Z_w\delta w + Z_v'\delta v' + Z_w'\delta w' + Z_\phi\delta\phi] d\eta\, d\zeta\, dx \tag{5.4}$$

where the terms $Z_u, Z_v, Z_w, Z_v', Z_w', Z_\phi$ are the coefficients of $\delta u, \delta v, \delta w, \delta v', \delta w', \delta\phi$ in the variation of kinetic energy expression.

Integration of the expression over the cross-section yields:

$$\delta T = m\Omega^2 l^3 \sum_i \int_0^{(l_e)_i} [\bar{Z}_u\delta u + \bar{Z}_v\delta v + \bar{Z}_w\delta w + \bar{Z}'_v\delta v' + \bar{Z}'_w\delta w' + \bar{Z}_\phi\delta\phi]\, dx \tag{5.5}$$

After eliminating the higher order terms, the expressions for $\bar{Z}_u, \bar{Z}_v, \bar{Z}_w, \bar{Z}'_v, \bar{Z}'_w, \bar{Z}_\phi$ are obtained.

5.2 STRAIN ENERGY OF BLADE

The formulation of strain energy outlined in this section essentially follows the prece-dure given in Refs. 24 and 28 .//

5.2.1 Strain Energy

The strain energy of a beam element is given by

$$U = \frac{1}{2} E_o l^3 \int_0^{l_e} \int \int \left\{ \begin{array}{c} \epsilon_{xx} \\ \gamma_{x\eta} \\ \gamma_{x\zeta} \end{array} \right\}^T \left\{ \begin{array}{c} \sigma_{xx} \\ \sigma_{x\eta} \\ \sigma_{x\zeta} \end{array} \right\} d\eta \, d\zeta \, dx \tag{5.6}$$

5.2.2 Explicit Strain-Displacement Relations

The expressions for non zero strain components written in terms of u,v,w and ϕ are given as [24, 27]

$$\begin{aligned}
\epsilon_{xx} &= \underline{u_x + \frac{1}{2} v_x^2 + \frac{1}{2} w_x^2} + \frac{1}{2}(\eta^2 + \zeta^2)\phi_x^2 + \alpha_x \Psi \\
&\quad + \alpha \tau_o (\zeta \Psi_\eta - \eta \Psi_\zeta) - [\eta \cos(\theta_g + \phi) - \zeta \sin(\theta_g + \phi)] v_{xx} \\
&\quad - [\eta \sin(\theta_g + \phi) + \zeta \cos(\theta_g + \phi)] w_{xx} + \eta(\bar{\gamma}_{x\eta,x} - \tau_o \bar{\gamma}_{x\zeta}) + \zeta(\bar{\gamma}_{x\zeta,x} + \tau_o \bar{\gamma}_{x\eta})
\end{aligned}$$

$$\gamma_{x\eta} = \bar{\gamma}_{x\eta} + \alpha \Psi_\eta - \zeta(\phi_x + \phi_o)$$

$$\gamma_{x\zeta} = \bar{\gamma}_{x\zeta} + \alpha \Psi_\zeta - \eta(\phi_x + \phi_o)$$

where

$$\phi_o = (v_{xx} \cos\theta_G + w_{xx} \sin\theta_G)(-v_x \sin\theta_G + w_x \cos\theta_G)$$

The underlined term in ϵ_{xx} represents the axial strain at the elastic axis. These strain expressions can be simplified using the following assumptions:

- The transverse shear at the elastic axis is assumed to be zero.
- The wraping amplitude α is assumed to be equal to -ϕ_x.

The simplified strain components can be written as

$$\begin{aligned}
\epsilon_{xx} &= u_x + \frac{1}{2}v_x^2 + \frac{1}{2}w_x^2 + \frac{1}{2}(\eta^2+\zeta^2)\phi_x^2 - \Psi\phi_{xx} - [\tau_o(\zeta\Psi_\eta - \eta\Psi_\zeta)]\psi_x \\
&\quad [\eta\cos(\theta_G+\phi_k) \; - \zeta \; \sin(\theta_G+\phi_k) \;]v_{xx} - [\eta\sin(\theta_G+\phi_k) \; + \zeta\cos(\theta_G+\phi_k) \;]w_{xx} \\
\gamma_{x\eta} &= -(\Psi_\eta + \zeta)\phi_x - \zeta\phi_o \\
\gamma_{x\zeta} &= -(\Psi_\zeta + \eta)\phi_x + \eta\phi_o \\
\phi_o &= (v_{xx}\cos\theta_G + w_{xx}\sin\theta_G)(-v_x\sin\theta_G + w_x\cos\theta_G)
\end{aligned}$$

5.2.3 Stress-Strain Relations

Assuming that the blade is made of isotropic material. The stress-strain relationship is given by the following equations:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{x\eta} \\ \sigma_{x\zeta} \end{Bmatrix} = \begin{bmatrix} E & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \epsilon_{xx} \\ \gamma_{x\eta} \\ \gamma_{x\zeta} \end{Bmatrix} \tag{5.7}$$

5.2.4 Strain Energy Variation

The variation of strain energy of the i^{th} element is given by

$$\delta U = E_o l^3 \int_0^{l_e} \int\int \begin{Bmatrix} \delta\epsilon_{xx} \\ \delta\gamma_{x\eta} \\ \delta\gamma_{x\zeta} \end{Bmatrix}^T \begin{Bmatrix} \sigma_{xx} \\ \sigma_{x\eta} \\ \sigma_{x\zeta} \end{Bmatrix} d\eta \, d\zeta \, dx \tag{5.8}$$

The variation of the strain components is given as:

$$
\begin{aligned}
\delta\epsilon_{xx} &= \delta u_x + v_x\delta v_x + w_x\delta w_x + (\eta^2+\zeta^2)\phi_x\delta\phi_{xx} - \Psi\delta\phi_{xx} - [\tau_o(\zeta\Psi_\eta - \eta\Psi_\zeta)]\delta\phi_x \\
&\quad -(\eta\cos\theta_G - \zeta\sin\theta_G)(\delta v_{xx} + \phi\delta w_{xx} + w_{xx}\delta\phi) \\
&\quad -(\eta\sin\theta_G + \zeta\cos\theta_G)(\delta w_{xx} - \phi\delta v_{xx} - v_{xx}\delta\phi)
\end{aligned}
$$

$$
\begin{aligned}
\delta\gamma_{x\eta} &= -(\Psi_\eta + \zeta)\delta\phi_x - \zeta\delta\phi_o \\
\delta\gamma_{x\zeta} &= -(\Psi_\zeta - \eta)\delta\phi_x + \eta\delta\phi_o
\end{aligned}
$$

where

$$
\begin{aligned}
\delta\phi_o &= (-\delta v_x\sin\theta_G + \delta w_x\cos\theta_G)(v_{xx}\cos\theta_G + w_{xx}\sin\theta_G) \\
&\quad +(-v_x\sin\theta_G + w_x\cos\theta_G)(\delta v_{xx}\cos\theta_G + \delta w_{xx}\sin\theta_G)
\end{aligned}
$$

It is assumed that the variations of strain components are of the same order as the corresponding strain components. Substituting the above expressions in the strain energy variation and integrating over the cross-section, the variation in strain energy for a beam element is obtained. The expression for δU_i, interms of stress and moment resultants, is given as:

$$
\begin{aligned}
\delta U_i &= E_o l^3 \int_0^{l_e} \{\bar{V}_x(\delta u_x + v_x\delta v_x + w_x\delta w_x) \\
&\quad +(-\tau_o\bar{P}'_x - \bar{R}_x + \bar{S}_x + \bar{T}_x\phi_x)\delta\phi_x + \bar{S}'_x\delta\phi_o - \bar{P}_x\delta\phi_{xx} \\
&\quad +[v_{xx}(\bar{M}'_\eta\cos\theta_G - \bar{M}'_\zeta\sin\theta_G) + w_{xx}(\bar{M}'_\eta\sin\theta_G + \bar{M}'_\zeta\cos\theta_G)]\delta\phi \\
&\quad +[(\bar{M}_\eta\sin\theta_G + \bar{M}_\zeta\cos\theta_G) + \phi(\bar{M}'_\eta\cos\theta_G - \bar{M}'_\zeta\sin\theta_G)]\delta v_{xx} \\
&\quad +[(-\bar{M}_\eta\cos\theta_G + \bar{M}_\zeta\sin\theta_G) + \phi(\bar{M}'_\eta\sin\theta_G + \bar{M}'_\zeta\cos\theta_G)]\delta w_{xx}\}dx
\end{aligned}
$$

Stress and Moment Resultants are defined as:

$$
\begin{aligned}
\bar{V}_x &= \int\int (E\epsilon_{xx}) d\eta d\zeta \\
&= E\,A[u_x + \frac{1}{2}v_x^2 + \frac{1}{2}w_x^2] \\
&\quad + \frac{1}{2}EAC_o[\phi_x^2] - EAD_o[\phi_{xx}] - \tau_o EAD_o'[\phi_x] \\
&\quad - (\overline{EA\eta_a} - \phi\overline{EA\zeta_a})[v_{xx}] - (\overline{EA\zeta_a} + \phi\overline{EA\eta_a})[w_{xx}]
\end{aligned}
\tag{5}
$$

$$
\begin{aligned}
\bar{M}_\eta &= \int\int (\zeta E\epsilon_{xx}) d\eta d\zeta \\
&= E\,A\zeta_a[u_x + \frac{1}{2}v_x^2 + \frac{1}{2}w_x^2] \\
&\quad + \frac{1}{2}EAC_2[\phi_x^2] - EAD_2[\phi_{xx}] - \tau_o EAD_2'[\phi_x] \\
&\quad - (\overline{EI_{\eta\zeta}} - \phi\overline{EI_{\eta\eta}})[v_{xx}] - (\overline{EI_{\eta\eta}} + \phi\overline{EI_{\eta\zeta}})[w_{xx}]
\end{aligned}
\tag{5.1}
$$

$$
\begin{aligned}
\bar{M}_\eta' &= \int\int (\zeta E\epsilon_{xx}) d\eta d\zeta \\
&= E\,A\zeta_a[u_x + \frac{1}{2}v_x^2 + \frac{1}{2}w_x^2] \\
&\quad - EAD_2[\phi_{xx}] - \tau_o EAD_2'[\phi_x] \\
&\quad - (\overline{EI_{\eta\zeta}})[v_{xx}] - (\overline{EI_{\eta\eta}})[w_{xx}]
\end{aligned}
\tag{5.12}
$$

$$
\begin{aligned}
\bar{M}_\zeta &= \int\int (-\eta E\epsilon_{xx}) d\eta d\zeta \\
&= -E\,A\eta_a[u_x + \frac{1}{2}v_x^2 + \frac{1}{2}w_x^2] \\
&\quad - \frac{1}{2}EAC_1[\phi_x^2] + EAD_1[\phi_{xx}] + \tau_o EAD_1'[\phi_x] \\
&\quad + (\overline{EI_{\zeta\zeta}} - \phi(\overline{EI_{\zeta\eta}})[v_{xx}] + (\overline{EI_{\zeta\eta}} + \phi\overline{EI_{\zeta\zeta}})[w_{xx}]
\end{aligned}
\tag{5.13}
$$

$$
\begin{aligned}
\bar{M}'_\zeta &= \int\int(-\eta E\epsilon_{xx})d\eta d\zeta \\
&= -E\,A\eta_a[u_x + \frac{1}{2}v_x^2 + \frac{1}{2}w_x^2] \\
&\quad +EAD_1[\phi_{xx}] + \tau_o EAD'_1[\phi_x] \\
&\quad +(\overline{EI_{\zeta\zeta}})[v_{xx}] + (\overline{EI_{\zeta\eta}})[w_{xx}]
\end{aligned}
\tag{5.14}
$$

$$
\begin{aligned}
\bar{P}_x &= \int\int(\Psi E\epsilon_{xx})d\eta d\zeta \\
&= +E\,AD_o[u_x + \frac{1}{2}v_x^2 + \frac{1}{2}w_x^2] \\
&\quad +\frac{1}{2}EAD_4[\phi_x^2] - EAD_3[\phi_{xx}] - \tau_o EAD_5[\phi_x] \\
&\quad -(\overline{EAD_1} - \phi\overline{EAD_2})[v_{xx}] - (\overline{EAD_2} + \phi\overline{EAD_1})[w_{xx}]
\end{aligned}
\tag{5.15}
$$

$$
\begin{aligned}
\bar{P}'_x &= \int\int[(\zeta\Psi_\eta - \eta\Psi_\zeta)E\epsilon_{xx})]d\eta d\zeta \\
&= +E\,AD'_o[u_x + \frac{1}{2}v_x^2 + \frac{1}{2}w_x^2] \\
&\quad +\frac{1}{2}EAD'_4[\phi_x^2] - EAD_5[\phi_{xx}] - \tau_o EAD'_3[\phi_x] \\
&\quad -(\overline{EAD'_1} - \phi\overline{EAD_2}')[v_{xx}] - (\overline{EAD_2}' + \phi\overline{EAD_1}')[w_{xx}]
\end{aligned}
\tag{5.16}
$$

$$
\begin{aligned}
\bar{T}_x &= \int\int[(\eta^2 - \zeta^2)E\epsilon_{xx})]d\eta d\zeta \\
&= +E\,AC_o[u_x + \frac{1}{2}v_x^2 + \frac{1}{2}w_x^2] \\
&\quad +\frac{1}{2}EAC_3[\phi_x^2] - EAD_4[\phi_{xx}] - \tau_o EAD'_4[\phi_x] \\
&\quad -(\overline{EAC'_1} - \phi\overline{EAC_2}')[v_{xx}] - (\overline{EAC_2}' + \phi\overline{EAC_1}')[w_{xx}]
\end{aligned}
\tag{5.17}
$$

$$
\begin{aligned}
\bar{R}_x &= \int\int[(\Psi_\eta\gamma_{x\eta} - \Psi_\zeta\gamma_{x\zeta})G]d\eta d\zeta \\
&= (GJ_1 = GJ_2)[\phi_x] + GJ_1[\phi_o]
\end{aligned}
\tag{5.18}
$$

$$\begin{aligned}
\bar{S}_x &= \int\int[(\eta\gamma_{x\zeta} - \zeta\gamma_{x\eta})G]d\eta d\zeta \\
&= (GJ_0 = GJ_1)[\phi_x] + GJ_1[\phi_o]
\end{aligned} \tag{5.19}$$

$$\begin{aligned}
\bar{S}'_x &= \int\int[(\eta\gamma_{x\zeta} - \zeta\gamma_{x\eta})G]d\eta d\zeta \\
&= (GJ_0 = GJ_1)[\phi_x]
\end{aligned} \tag{5.20}$$

In the above expressions, the moment resultants $\bar{M}'_\eta, \bar{M}'_\zeta$ and $\bar{S}'_x$ have the same definitions as $\bar{M}_\eta, \bar{M}_\zeta$ and $\bar{S}_x$ respectively. But higher order terms are neglected in defining $\bar{M}'_\eta, \bar{M}'_\zeta$ and $\bar{S}'_x$, since these expressions are coupled with terms of order ϵ^2. Whereas $\bar{M}_\eta, \bar{M}_\zeta$ and $\bar{S}_x$ are coupled with terms of order ϵ. The cross-sectional integrals associated with the strain energy variation are defined as:

$$\begin{aligned}
EA &= \int\int[E]d\eta d\zeta \\
EA\eta_a &= \int\int[E\eta]d\eta d\zeta \\
EA\zeta_a &= \int\int[E\zeta]d\eta d\zeta \\
EI_{\eta\eta} &= \int\int[E\zeta^2]d\eta d\zeta \\
EI_{\zeta\zeta} &= \int\int[E\eta^2]d\eta d\zeta \\
EI_{\eta\zeta} &= \int\int[E\eta\zeta]d\eta d\zeta \\
EAC_o &= \int\int[E(\eta^2 + \zeta^2)]d\eta d\zeta \\
EAC_1 &= \int\int[E\eta(\eta^2 + \zeta^2)]d\eta d\zeta \\
EAC_2 &= \int\int[E\zeta(\eta^2 + \zeta^2)]d\eta d\zeta \\
EAC_3 &= \int\int[E(\eta^2 + \zeta^2)^2]d\eta d\zeta \\
EAD_o &= \int\int[E\Psi]d\eta d\zeta \\
EAD_1 &= \int\int[E\eta\Psi]d\eta d\zeta
\end{aligned}$$

$$\begin{aligned}
EAD_2 &= \int\int[E\zeta\Psi]d\eta d\zeta \\
EAD_3 &= \int\int[E\Psi^2]d\eta d\zeta \\
EAD_4 &= \int\int[E(\eta^2+\zeta^2)\Psi]d\eta d\zeta \\
EAD_5 &= \int\int[E(\zeta\Psi_\eta-\eta\Psi_\zeta)\Psi]d\eta d\zeta \\
EAD'_o &= \int\int[E(\zeta\Psi_\eta-\eta\Psi_\zeta)]d\eta d\zeta \\
EAD'_1 &= \int\int[E\eta(\zeta\Psi_\eta-\eta\Psi_\zeta)]d\eta d\zeta \\
EAD'_2 &= \int\int[E\zeta(\zeta\Psi_\eta-\eta\Psi_\zeta)]d\eta d\zeta \\
EAD'_3 &= \int\int[E(\zeta\Psi_\eta-\eta\Psi_\zeta)^2]d\eta d\zeta \\
EAD'_4 &= \int\int[E(\eta^2+\zeta^2)(\zeta\Psi_\eta-\eta\Psi_{zeta})]d\eta d\zeta \\
GJ_o &= \int\int[G(\eta^2+\zeta^2)]d\eta d\zeta \\
GJ_1 &= \int\int[G(\zeta\Psi_\eta-\eta\Psi_\zeta)]d\eta d\zeta \\
GJ_2 &= \int\int[G(\Psi_\eta^2+\Psi_\zeta^2)]d\eta d\zeta
\end{aligned}$$

Certain additional terms are defined as:

$$\begin{aligned}
\overline{EA\eta_a} &= EA\eta_a\cos\theta_G - EA\zeta_a\sin\theta_G \\
\overline{EA\zeta_a} &= EA\eta_a\sin\theta_G + EA\zeta_a\cos\theta_G \\
\overline{EAC_1} &= EAC_1\cos\theta_G - EAC_2\sin\theta_G \\
\overline{EAC_2} &= EAC_1\sin\theta_G + EAC_2\cos\theta_G \\
\overline{EAD_1} &= EAD_1\cos\theta_G - EAD_2\sin\theta_G \\
\overline{EAD_2} &= EAD_1\sin\theta_G + EAD_2\cos\theta_G \\
\overline{EAD'_1} &= EAD'_1\cos\theta_G - EAD'_2\sin\theta_G \\
\overline{EAD'_2} &= EAD'_1\sin\theta_G + EAD'_2\cos\theta_G
\end{aligned}$$

$$
\begin{aligned}
\overline{EI_{\eta\zeta}} &= EI_{\eta\zeta}\cos\theta_G - EI_{\eta\eta}\sin\theta_G \\
\overline{EI_{\eta\eta}} &= EI_{\eta\zeta}\sin\theta_G + EI_{\eta\eta}\cos\theta_G \\
\overline{EI_{\zeta\zeta}} &= EI_{\zeta\zeta}\cos\theta_G - EI_{\zeta\eta}\sin\theta_G \\
\overline{EI_{\zeta\eta}} &= EI_{\zeta\zeta}\sin\theta_G + EI_{\zeta\eta}\cos\theta_G \\
GJ &= GJ_o - 2GJ_1 + GJ_2
\end{aligned}
$$

Chapter 6

FORMULATION OF ELEMENT MATRICES ASSOCIATED WITH KINETIC AND STRAIN ENERGY VARIATION

6.1 FINITE ELEMENT DISCRETIZATION

The variational expressions associated with the kinetic and potential energy of the rotor blades are nonlinear. The unknowns are the deformation functions u,v,w, and ϕ. These are dependent on both space and time. The spatial dependence is eliminated using a Rayleigh Ritz finite element formulation. The blade is divided into subregions (finite elements) as shown in Fig. 10, and the total dynamic potential is calculated for each subregion. By applying Hamilton's principle to each subregion, a discretized form of the equations of motion can be obtained. In this development, each subregion is modelled by a straight beam type finite element. These beam elements are located along the elastic axis (line of shear centers) of the blade.

The discretized form of Hamilton's principle is written as:

$$\int_{t_1}^{t_2} \sum_{i=1}^{N} (\delta U_i - \delta T_i - \delta W_{ei}) dt = 0 \tag{6.1}$$

where:

N= Total number of finite elements in the model.

δU_i =Variation of the strain energy in the i^{th} element.

δT_i =Variation of the kinetic energy in the i^{th} element.

δW_{ei} =Virtual work of the external loads in i^{th} element.

Assume that the four unknown displacement functions of the beam finite element are expressed in the following form.

$$\begin{bmatrix} v \\ w \\ \phi \\ u \end{bmatrix} = \begin{bmatrix} \{\phi_v\}^T & 0 & 0 & 0 \\ 0 & \{\phi_w\}^T & 0 & 0 \\ 0 & 0 & \{\phi_\phi\}^T & 0 \\ 0 & 0 & 0 & \{\phi_u\}^T \end{bmatrix} \begin{bmatrix} \{V\} \\ \{W\} \\ \{\Phi\} \\ \{U\} \end{bmatrix} \tag{6.2}$$

where $\{\phi_v\}, \{\phi_w\}, \{\phi_\phi\}, \{\phi_u\}$ are space dependent interpolation functions; and $\{V\}, \{W\}, \{\Phi\}, \{U\}$ are time dependent nodal values of v, w, ϕ, u, respectively.

$$\{V\} = \begin{Bmatrix} v_1 \\ v_1' \\ v_2 \\ v_2' \end{Bmatrix}; \{W\} = \begin{Bmatrix} w_1 \\ w_1' \\ w_2 \\ w_2' \end{Bmatrix}; \{\Phi\} = \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix}; \{U\} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

The nodal coordinates are shown in Fig.11.

The variation of the displacement functions for the beam can be written as:

$$\begin{bmatrix} \delta v \\ \delta w \\ \delta\phi \\ \delta u \end{bmatrix} = \begin{bmatrix} \{\phi_v\}^T & 0 & 0 & 0 \\ 0 & \{\phi_w\}^T & 0 & 0 \\ 0 & 0 & \{\phi_\phi\}^T & 0 \\ 0 & 0 & 0 & \{\phi_u\}^T \end{bmatrix} \begin{bmatrix} \{\delta V\} \\ \{\delta W\} \\ \{\delta\Phi\} \\ \{\delta U\} \end{bmatrix} \tag{6.3}$$

In this development, Hermite interpolation polynomials are used for the space dependent interpolation functions. A cubic Hermite interpolation polynomial, $\{\Phi_c\}$, is

used for the bending deflections (v,w) and a quadratic Hermite interpolation polynomial, $\{\Phi_q\}$, is used for the torsional rotation (ϕ), and the axial deflection (u). The mathematical expressions for these polynomials are given as:

$$\{\phi_v\} = \{\phi_w\} = \begin{bmatrix} 1-3\xi^2+2\xi^3 \\ l_e(\xi-2\xi^2+\xi^3) \\ 3\xi^2-2\xi^3 \\ l_e(-\xi^2+\xi^3) \end{bmatrix} = \{\Phi_c\} \tag{6.4}$$

$$\{\phi_\phi\} = \{\phi_u\} = \begin{bmatrix} 1-3\xi+2\xi^2 \\ 4\xi-4\xi^2 \\ -\xi+\xi^2 \end{bmatrix} = \{\Phi_q\} \tag{6.5}$$

where:

$\xi = x_k/l_e$

x_k = spanwise (axial) coordinate of the beam element.

l_e = length of beam element.

For bending deformations, the nodal parameters are the displacements and slopes at both ends of the beam element. Therefore, the resulting elements will have interelement continuity for both displacements and slopes. In addition, because of the cubic interpolation polynomial, bending strains vary linearly over the element length.

The quadratic Hermite interpolation polynomial is used for torsional rotation (ϕ) and the deflection (u). This polynomial has the capability of modelling a linear variation of strains along the element length and therefore provides the same lavel of accuracy as the beam bending element. The nodal parameters for these elements are chosen as the values of the displacements function at the two end nodes and at the mid-point of the element.

The resulting beam element has 14 degree of freedom: 4 in-plane (lag) bending degrees of freedom, 4 out-plane (flap) degrees of freedom, and 3 degrees of freedom each of torsion (ϕ), and axial deflection (u). The nodal degrees of freedom are shown in Fig.11.

6.2 ELEMENT MATRICES ASSOCIATED WITH KINETIC ENERGY VARIATION

The beam element matrices associated with the kinetic energy variation are derived by substituting the assumed expressions for the displacements funtions (Eq. 6.2) in the kinetic energy variation δT (Eq. 5.5) and carrying out the intergration over the length of the beam element. The resulting variation of the kinetic energy can be written in the form:

$$\begin{aligned}
\delta T_i \; = \; & -\{\delta q\}^T([M]_{14\times14}\{\ddot{q}\} + [M^C]_{14\times14}\{\dot{q}\} + [K^{cf}]^{14\times14}\{q\} + \{V^L\}_{14\times1} \\
& +[M^1]_{14\times3}\left\{\begin{array}{c}\ddot{R}_x\\ \ddot{R}_y\\ \ddot{R}_z\end{array}\right\} + [M^2]_{14\times3}\left\{\begin{array}{c}\dot{R}_x\\ \dot{R}_y\\ \dot{R}_z\end{array}\right\} + [M^3]_{14\times3}\left\{\begin{array}{c}\ddot{\theta}_x\\ \ddot{\theta}_y\\ \ddot{\theta}_z\end{array}\right\} + [M^4]_{14\times3}\left\{\begin{array}{c}\dot{\theta}_x\\ \dot{\theta}_y\\ \dot{\theta}_z\end{array}\right\} \\
& +\{V^I\}_{14\times1} + \{V^{NL}\}_{14\times1})
\end{aligned} \tag{6.6}$$

Where $\{q\}$ represents the vector of unknown nodal degrees of freedom

$$\{q\}_{14\times1} = \left\{\begin{array}{c}\{V\}\\ \{W\}\\ \{\Phi\}\\ \{U\}\end{array}\right\} \tag{6.7}$$

Detailed expressions for the various matrices defined in Eq. 6.6 are given as follows:

6.2.1 Mass Matrix $[M]_{14\times 14}$

$$
\begin{aligned}
[M_{11}] = \int_0^{l_e} & [(\cos^2\Lambda_s \ + \sin\Lambda_s \ \cos\Lambda_a \ + \sin\Lambda_s \ \sin\Lambda_a) m\{\Phi_c\} \ \{\Phi_c\}^T \\
& -(\sin\Lambda_s \ \cos\Lambda_a \ \cos\Lambda_s \ + \cos\Lambda_s \ \sin\Lambda_s \ \cos^2\Lambda_a \ - \sin\Lambda_a \ \sin\Lambda_s \ \cos\Lambda_a \ ) \\
& \{m\eta_m \cos(\theta_G+\phi_k) - m\zeta_m \sin(\theta_G+\phi_k)\} \ \{\Phi_c\} \ \{\Phi_c'\}^T \\
& -(\cos\Lambda_s \ \cos^2\Lambda_a \ \sin\Lambda_s \ + \sin\Lambda_s \ \cos\Lambda_a \ \cos\Lambda_s \ - \sin\Lambda_s \ \sin^2\Lambda_a \ ) \\
& \{m\eta_m \cos(\theta_G+\phi_k) - m\zeta_m \sin(\theta_G+\phi_k)\} \ \{\Phi_c'\} \ \{\Phi_c\}^T \\
& +(\cos^2\Lambda_s \ \cos^2\Lambda_a \ + \sin^2\Lambda_s \ \cos^2\Lambda_a \ + \sin^2\Lambda_a \ ) \\
& (\{Im_{\zeta\zeta}\cos^2(\theta_G+\phi_k) + Im_{\eta\eta}\sin^2(\theta_G+\phi_k)\} \\
& -\{I_{\eta\eta} - I_{\zeta\zeta}\}\sin(\theta_G+\phi_k)\cos(\theta_G+\phi_k) \ )\{\Phi_c'\} \ \{\Phi_c'\}^T \ ]dx
\end{aligned}
$$

$$
\begin{aligned}
[M_{12}] = \int_0^{l_e} & [(-\cos\Lambda_s \ \sin\Lambda_s \ \cos\Lambda_a \ - \sin\Lambda_s \ \cos^2\Lambda_a \ \cos\Lambda_s \ + \sin\Lambda_s \ \sin^2\Lambda_a \ ) \\
& \{m\eta_m \sin(\theta_G+\phi_k) + m\zeta_m \cos(\theta_G+\phi_k)\} \ \{\Phi_c\} \ \{\Phi_c'\}^T \\
& +(\cos\Lambda_s \ \cos\Lambda_a \ \sin\Lambda_a \ + \sin\Lambda_a \ \cos\Lambda_a \ ) \\
& \{m\eta_m \cos(\theta_G+\phi_k) - m\zeta_m \sin(\theta_G+\phi_k)\} \ \{\Phi_c'\} \ \{\Phi_c\}^T \\
& +(\cos^2\Lambda_s \ \cos^2\Lambda_a \ + \sin^2\Lambda_s \ \cos^2\Lambda_a \ + \sin^2\Lambda_a \ ) \\
& (\{I_{\eta\eta} - I_{\zeta\zeta}\}\sin(\theta_G+\phi_k)\cos(\theta_G+\phi_k) \ + I_{\eta\zeta}\{\cos^2(\theta_G+\phi_k) - \sin^2(\theta_G+\phi_k)\} \ ) \\
& \{\Phi_c'\} \ \{\Phi_c'\}^T \ ]dx
\end{aligned}
$$

$$
\begin{aligned}
[M_{13}] = \int_0^{l_e} & [(-\cos^2\Lambda_s \ + \sin^2\Lambda_s \ \cos\Lambda_a \ ) \\
& \{m\eta_m \sin(\theta_G+\phi_k) + m\zeta_m \cos(\theta_G+\phi_k)\} \ \{\Phi_c\} \ \{\Phi_q\}^T \\
& +(\cos\Lambda_s \ \sin\Lambda_s \ \sin\Lambda_a \ + \sin\Lambda_s \ \cos\Lambda_a \ \cos\Lambda_s \ \sin\Lambda_a \ + \sin\Lambda_s \ \sin\Lambda_a \ \cos\Lambda_a \ ) \\
& \{m\eta_m \cos(\theta_G+\phi_k) - m\zeta_m \sin(\theta_G+\phi_k)\} \ \{\Phi_c\} \ \{\Phi_q\}^T \\
& (-\cos^2\Lambda_s \ \sin\Lambda_a \ \cos\Lambda_a \ - \sin^2\Lambda_s \ \cos\Lambda_a \ \sin\Lambda_a \ + \cos\Lambda_a \ \sin\Lambda_a \ ) \\
& (\{Im_{\zeta\zeta}\cos^2(\theta_G+\phi_k) + Im_{\eta\eta}\sin^2(\theta_G+\phi_k)\} \ + \{2I_{\eta\zeta}\cos(\theta_G+\phi_k)\sin(\theta_G+\phi_k)\} \ )
\end{aligned}
$$

$$\{\Phi_c'\}\ \{\Phi_q\}^T\ ]dx$$

$$\begin{aligned}
[M_{14}] &= \int_0^{l_e} [(-\cos\Lambda_s\ \sin\Lambda_s\ + \sin\Lambda_s\ \cos\Lambda_s\ \cos^2\Lambda_a \\
&\quad + \cos\Lambda_s\ \sin^2\Lambda_a\ \sin\Lambda_s\ )m\{\Phi_c\}\ \{\Phi_q\}^T \\
&\quad +(-\cos^2\Lambda_s\ \cos^2\Lambda_a\ + \sin^2\Lambda_s\ \cos\Lambda_a\ + \sin\Lambda_s\ \sin\Lambda_a\ \cos\Lambda_a\ ) \\
&\quad \{m\eta_m\cos(\theta_G+\phi_k) - m\zeta_m\sin(\theta_G+\phi_k)\}\ \{\Phi_c'\}\ \{\Phi_q\}^T\ ]dx
\end{aligned}$$

$$\begin{aligned}
[M_{21}] &= [M_{12}]^T \\
[M_{22}] &= [(\cos^2\Lambda_a\ - \sin\Lambda_s\ \sin\Lambda_a\ )m\{\Phi_c\}\ \{\Phi_c\}^T \\
&\quad (\cos\Lambda_a\ \sin\Lambda_a\ + \sin\Lambda_s\ \cos\Lambda_s\ \cos\Lambda_a\ ) \\
&\quad \{m\eta_m\cos(\theta_G+\phi_k) - m\zeta_m\sin(\theta_G+\phi_k)\}\ \{\Phi_c\}\ \{\Phi_c'\}^T \\
&\quad \{m\eta_m\sin(\theta_G+\phi_k) + m\zeta_m\cos(\theta_G+\phi_k)\}\ \{\Phi_c\}\ \{\Phi_c'\}^T \\
&\quad (\cos^2\Lambda_s\ \cos^2\Lambda_a\ + \sin^2\Lambda_s\ \cos^2\Lambda_a\ + \sin^2\Lambda_a\ ) \\
&\quad (\{Im_{\zeta\zeta}\sin^2(\theta_G+\phi_k) + Im_{\eta\eta}\cos^2(\theta_G+\phi_k)\}\ + \{2I_{\eta\zeta}\cos(\theta_G+\phi_k)\sin(\theta_G+\phi_k)\}\ ) \\
&\quad \{\Phi_c'\}\ \{\Phi_c'\}^T\ ]dx
\end{aligned}$$

$$\begin{aligned}
[M_{23}] &= \int_0^{l_e} (\cos^2\Lambda_a\ - \sin\Lambda_s\ \sin\Lambda_a\ \cos\Lambda_s\ ) \\
&\quad \{m\eta_m\cos(\theta_G+\phi_k) - m\zeta_m\sin(\theta_G+\phi_k)\}\ \{\Phi_c\}\ \{\Phi_q\}^T \\
&\quad - \sin^2\Lambda_s\ \{m\eta_m\sin(\theta_G+\phi_k) + m\zeta_m\cos(\theta_G+\phi_k)\}\ \{\Phi_c\}\ pqt \\
&\quad (\cos^2\Lambda_s\ \cos\Lambda_a\ \sin\Lambda_a\ + \sin^2\Lambda_s\ \sin\Lambda_a\ \cos\Lambda_a\ - \sin\Lambda_a\ \cos\Lambda_a\ ) \\
&\quad (\{I_{\eta\eta} - I_{\zeta\zeta}\}\sin(\theta_G+\phi_k)\cos(\theta_G+\phi_k)\ + I_{\eta\zeta}\{\cos^2(\theta_G+\phi_k) - \sin^2(\theta_G+\varphi_k)\}\ ) \\
&\quad \{\Phi_c'\}\ \{\Phi_q\}^T\ ]dx
\end{aligned}$$

$$M_{24}] = \int_0^{l_e} [(\cos\Lambda_s\ \cos\Lambda_a\ \sin\Lambda_a\ - \sin\Lambda_s\ \cos\Lambda_s\ \cos\Lambda_a\ )$$

$$m\{\Phi_c\}\ \{\Phi_q\}^T\ + (-\cos^2\Lambda_s\ \cos\Lambda_a\ + \sin^2\Lambda_s\ \cos\Lambda_a\ + \sin^2\Lambda_a\ \cos\Lambda_s\ )$$
$$\{m\eta_m \sin(\theta_G+\phi_k) + m\zeta_m \cos(\theta_G+\phi_k)\}\ \{\Phi_c'\}\ \{\Phi_q\}^T\ ]dx$$

$$[M_{31}] \ = \ [M_{13}]^T$$

$$[M_{32}] \ = \ [M_{23}]^T$$

$$\begin{aligned}
[M_{33}] \ = \ & \int_0^{l_e} [(\cos^2\Lambda_a\ + \cos^2\Lambda_s\ \sin^2\Lambda_a\ - \sin^2\Lambda_s\ \sin^2\Lambda_a\ ) \\
& (Im_{\zeta\zeta} + I_{\eta\eta})\{\Phi_q\}\ \{\Phi_q\}^T \\
& (2\sin\Lambda_s\ \sin\Lambda_a\ \cos\Lambda_s\ )(I_{\eta\zeta}\{\cos^2(\theta_G+\phi_k) - \sin^2(\theta_G+\phi_k)\} \\
& -\{I_{\eta\eta} - I_{\zeta\zeta}\}\sin(\theta_G+\phi_k)\cos(\theta_G+\phi_k)\ )\{\Phi_q\}\ \{\Phi_q\}^T\ ]dx
\end{aligned}$$

$$\begin{aligned}
[M_{34}] \ = \ & \int_0^{l_e} [(\cos^2\Lambda_s\ \cos\Lambda_a\ \sin\Lambda_s\ + \sin^2\Lambda_s\ \cos\Lambda_a\ \cos\Lambda_s\ ) \\
& \{m\eta_m \sin(\theta_G+\phi_k) + m\zeta_m \cos(\theta_G+\phi_k)\}\ \{\Phi_q\}\ \{\Phi_q\}^T \\
& +(\cos\Lambda_s\ \cos\Lambda_a\ \sin^2\Lambda_a\ + \cos^2\Lambda_s\ \cos\Lambda_a\ \sin\Lambda_a\ ) \\
& \{m\eta_m \cos(\theta_G+\phi_k) - m\zeta_m \sin(\theta_G+\phi_k)\}\ \{\Phi_q\}\ \{\Phi_q\}^T\ ]dx
\end{aligned}$$

$$[M_{41}] \ = \ [M_{14}]^T$$

$$[M_{42}] \ = \ [M_{24}]^T$$

$$[M_{43}] \ = \ [M_{34}]^T$$

$$[M_{44}] \ = \ \int_0^{l_e} (\cos^2\Lambda_s\ \cos^2\Lambda_a\ - \sin^2\Lambda_s\ + \cos^2\Lambda_s\ \sin\Lambda_s\ \sin\Lambda_a\ )m\{\Phi_q\}\ \{\Phi_q\}^T\ dx$$

6.2.2 Matrix $[M^c]_{14\times 14}$

$$
\begin{aligned}
[M^c_{11}] &= \int_0^{l_e} m[-\sin\Lambda_s\ \cos\Lambda_a\ \cos\Lambda_s\ \ -\cos\Lambda_s\ \sin\Lambda_s\ \cos\Lambda_a \\
&\quad +\sin\Lambda_s\ \sin\Lambda_a\ \cos\Lambda_s\ ]\{\Phi_c\}\ \{\Phi_c\}^T \\
&\quad +[-m\eta_m\ \cos(\theta_G+\phi_k)\ +m\zeta_m\ \sin(\theta_G+\phi_k)\ ]\{\cos\Lambda_s\ \cos\Lambda_a \\
&\quad +\sin\Lambda_s\ \sin\Lambda_a\ \cos\Lambda_a\ \}\{\Phi_c\}\ \{\Phi'_c\}^T\ dx
\end{aligned}
$$

$$
[M^c_{12}] = \int_0^{l_e} m\ \cos\Lambda_s\ \cos\Lambda_a\ \{\Phi_c\}\ \{\Phi_c\}^T\ dx
$$

$$
\begin{aligned}
[M^c_{13}] &= \int_0^{l_e} [2\cos\Lambda_s\ \cos\Lambda_a\ \ (I_{\eta\eta}\ +I_{\zeta\zeta}\ )\sin(\theta_G+\phi_k)\ \cos(\theta_G+\phi_k)\ \{\Phi'_c\}\ \{\Phi_q\}^T \\
&\quad -\cos\Lambda_a\ \cos\Lambda_s\ I_{\eta\zeta}\ (\cos^2(\theta_G+\phi_k)\ -\sin^2(\theta_G+\phi_k)\ )]dx
\end{aligned}
$$

$$
[M^c_{14}] = 0
$$

$$
\begin{aligned}
[M^c_{21}] &= \int_0^{l_e} [\cos\Lambda_s\ \cos\Lambda_a\ \{(m\eta_m\ \cos(\theta_G+\phi_k)\ -m\zeta_m\ \sin(\theta_G+\phi_k)\ ) \\
&\quad -(m\eta_m\ \sin(\theta_G+\phi_k)\ +m\zeta_m\ \cos(\theta_G+\phi_k)\ )\}((\beta_s\cos\theta_I-\beta_d\sin\theta_I+1)\ )\{\Phi_c\}\ \{\Phi'_c\}^T \\
&\quad +\cos\Lambda_a\ \cos\Lambda_s\ \ \{m\eta_m\ \sin(\theta_G+\phi_k)\ -m\zeta_m\ \cos(\theta_G+\phi_k)\ \}\{\Phi_c\}\ \{\Phi'_c\}^T\ ]dx
\end{aligned}
$$

$$
\begin{aligned}
[M^c_{22}] &= \int_0^{l_e} [\cos\Lambda_s\ \cos\Lambda_a\ \{m\eta_m\ \sin(\theta_G+\phi_k)\ +m\zeta_m\ \cos(\theta_G+\phi_k)\ \} \\
&\quad ((\beta_s\cos\theta_I-\beta_d\sin\theta_I+1)\ )\ \{\Phi_c\}\ \{\Phi_c\}^T\ ]dx
\end{aligned}
$$

$$
[M^c_{23}] = \int_0^{l_e} [-\{m\eta_m\ \sin(\theta_G+\phi_k)\ +m\zeta_m\ \cos(\theta_G+\phi_k)\ \}\ \cos\Lambda_a\ \{\Phi_c\}\ \{\Phi_q\}^T
$$

$$-\sin\Lambda_s \ \ \sin\Lambda_a \ \{m\eta_m \ \sin(\theta_G+\phi_k) \ +m\zeta_m \ \cos(\theta_G+\phi_k) \ \}\{\Phi_c'\} \ \{\Phi_q\}^T$$

$$-\cos\Lambda_s \ \ \sin^2\Lambda_a \ \{m\eta_m \ \sin(\theta_G+\phi_k) \ +m\zeta_m \ \cos(\theta_G+\phi_k) \ \}\{\Phi_c'\} \ \{\Phi_q\}^T$$

$$-\cos\Lambda_s \ \ \cos\Lambda_a \ (I_{\zeta\zeta} \ -I_{\eta\eta} \ )(\sin^2(\theta_G+\phi_k) \ -\cos^2(\theta_G+\phi_k) \ )\{\Phi_c'\} \ \{\Phi_q\}^T$$

$$+\{(I_{\zeta\zeta} \ -I_{\eta\eta} \ -2I_{\eta\zeta} \ \}\sin(\theta_G+\phi_k) \ \cos(\theta_G+\phi_k) \ \ \cos^2\Lambda_s \ \cos^2\Lambda_a \ \{\Phi_c'\} \ \{\Phi_q\}^T \ ]dx$$

$$[M_{24}^c] = 0$$

$$[M_{31}^c] = \int_0^{l_e} \cos\Lambda_s \ \cos\Lambda_a \ [\{I_{\zeta\zeta} \ \cos(\theta_G+\phi_k) \ -I_{\eta\zeta} \ \sin(\theta_G+\phi_k) \ \}$$

$$(-\sin(\theta_G+\phi_k) \ \cos\Lambda_s \ +\cos(\theta_G+\phi_k) \ \sin\Lambda_s \ \sin\Lambda_a \ )+\{-I_{\eta\zeta} \ \cos(\theta_G+\phi_k)$$

$$+I_{\eta\eta} \ \sin(\theta_G+\phi_k) \ \}(-\cos\Lambda_s \ \cos(\theta_G+\phi_k) \ -\sin(\theta_G+\phi_k) \ \sin\Lambda_s \ \sin\Lambda_a \ )]$$

$$-\cos^2\Lambda_s \ \cos\Lambda_a \ \ [\{I_{\zeta\zeta} \ -I_{\eta\eta} \ \}\cos(\theta_G+\phi_k) \ \sin(\theta_G+\phi_k)$$

$$+I_{\eta\zeta} \ (\cos(\theta_G+\phi_k)2 \ -\sin^2(\theta_G+\phi_k) \ )]\{\Phi_q\} \ \{\Phi_c'\}^T \ dx$$

$$[M_{32}^c] = \int_0^{l_e} - \ \sin\Lambda_a \ [\{I_{\zeta\zeta} \ \sin(\theta_G+\phi_k)$$

$$+I_{\eta\zeta} \ \cos(\theta_G+\phi_k) \ \}(\sin(\theta_G+\phi_k) \ \sin\Lambda_s \ +\cos\Lambda_s \ \sin\Lambda_a \ \cos(\theta_G+\phi_k) \ )$$

$$+\{I_{\eta\zeta} \ \sin(\theta_G+\phi_k) \ +I_{\eta\eta} \ \cos(\theta_G+\phi_k) \ \}$$

$$(\cos(\theta_G+\phi_k) \ \sin\Lambda_s \ -\cos\Lambda_s \ \sin\Lambda_a \ \sin(\theta_G+\phi_k) \ )]-(\ \cos^2\Lambda_s \ \cos\Lambda_a$$

$$+ \ \sin^2\Lambda_s \ \cos\Lambda_a \ )\{I_{\zeta\zeta} \ \sin^2(\theta_G+\phi_k) \ +I_{\eta\eta} \ \cos^2(\theta_G+\phi_k)$$

$$+2I_{\eta\zeta} \ \sin(\theta_G+\phi_k) \ \cos(\theta_G+\phi_k) \ \}- \ \cos\Lambda_s \ \cos\Lambda_a \ [\{I_{\zeta\zeta} \ \sin(\theta_G+\phi_k)$$

$$+I_{\eta\zeta} \ \cos(\theta_G+\phi_k) \ \}(-\cos\Lambda_s \ \sin(\theta_G+\phi_k) \ +\sin\Lambda_s \ \sin\Lambda_a \ \cos(\theta_G+\phi_k) \ )$$

$$+\{I_{\eta\eta} \ \cos(\theta_G+\phi_k) \ +I_{\eta\zeta} \ \sin(\theta_G+\phi_k) \ \}(\cos\Lambda_s \ \cos(\theta_G+\phi_k)$$

$$-\sin\Lambda_s \ \sin\Lambda_a \ \sin(\theta_G+\phi_k) \ )]\{\Phi_c\} \ \{\Phi_c'\}^T \ dx$$

$$[M_{33}^c] = \int_0^{l_e} \cos\Lambda_s \ [\{I_{\zeta\zeta} \ \sin(\theta_G+\phi_k) \ +I_{\eta\zeta} \ \cos(\theta_G+\phi_k) \ \}$$

$$(\sin(\theta_G+\phi_k)\ \sin\Lambda_s\ +\cos(\theta_G+\phi_k)\ \cos\Lambda_s\ \sin\Lambda_a\ )+\{I_{\eta\zeta}\ \sin(\theta_G+\phi_k)$$
$$+I_{\eta\eta}\ \cos(\theta_G+\phi_k)\ \}(\cos(\theta_G+\phi_k)\ \sin\Lambda_s\ -\sin(\theta_G+\phi_k)\ \cos\Lambda_s\ \sin\Lambda_a\ )]$$
$$\{\Phi_q\}\ \{\Phi_q\}^T\ dx$$

$$[M^c_{34}] = \int_0^{l_e} [-m\ \cos\Lambda_s\ \cos\Lambda_a\ +\{m\eta_m\ \cos(\theta_G+\phi_k)\ -m\zeta_m\ \sin(\theta_G+\phi_k)\ \}$$
$$(\cos\Lambda_s\ -\sin\Lambda_s\ \sin\Lambda_a\ )-\{m\eta_m\ \sin(\theta_G+\phi_k)\ -m\zeta_m\ \cos(\theta_G+\phi_k)\ \}$$
$$\cos\Lambda_s\ \cos\Lambda_a\ ]\{\Phi_q\}\ \{\Phi_q\}^T\ dx$$

$$[M^c_{41}] = \int_0^{l_e} m[\cos^2\Lambda_s\ \cos\Lambda_a\ +\sin^2\Lambda_s\ \cos\Lambda_a$$
$$-\cos^2\Lambda_s\ \sin\Lambda_a\ ]\{\Phi_q\}\ \{\Phi_c\}^T\ dx$$

$$[M^c_{42}] = \int_0^{l_e} m[\cos^2\Lambda_s\ \cos\Lambda_a\ -\sin\Lambda_s\ \sin\Lambda_a$$
$$-\cos\Lambda_a\ \sin\Lambda_s\ ]\{\Phi_q\}\ \{\Phi_c\}^T\ dx$$

$$[M^c_{43}] = \int_0^{l_e} -\ \cos\Lambda_s\ \cos\Lambda_a\ (m\eta_m\ \cos(\theta_G+\phi_k)\ -m\zeta_m\ \sin(\theta_G+\phi_k)\ )\{\Phi_q\}\ \{\Phi'_c\}^T\ dx$$

$$[M^c_{44}] = \int_0^{l_e} m[-\ \cos^2\Lambda_s\ \sin\Lambda_a\ \cos\Lambda_a$$
$$+\sin\Lambda_s\ \sin\Lambda_a\ \cos\Lambda_s\ ]\{\Phi_q\}\ \{\Phi_q\}^T\ dx$$

6.2.3 Centrifugal Stiffness Matrix $[K^{cf}_{14\times 14}]$

$$
\begin{aligned}
[K_{11}] &= \int_0^{l_e} [-m(\cos^2\Lambda_a \ \cos^2\theta_I \\
&+\cos^2\theta_I \ \cos^2\Lambda_s \ \sin^2\Lambda_a \\
&-\cos\theta_I \ \sin^2\Lambda_s \ \sin\Lambda_a \ \cos\Lambda_a \ \beta_p + \beta_d\cos\theta_I - \beta_s\sin\theta_I \ ) \\
&\{\Phi_c\} \ pct \ + (\cos^2\theta_I \ \sin^2\Lambda_s \ \cos^2\Lambda_a \\
&+\cos^2\theta_I \ \cos^2\Lambda_s \ \cos^2\Lambda_a \ )(\{Im_{\zeta\zeta}\cos^2(\theta_G+\phi_k) + Im_{\eta\eta}\sin^2(\theta_G+\phi_k)\} \\
&-\{2I_{\eta\zeta}\cos(\theta_G+\phi_k)\sin(\theta_G+\phi_k)\} \ )\{\Phi_c'\} \ \{\Phi_c'\}^T \ ]dx
\end{aligned}
$$

$$
\begin{aligned}
[K_{12}] &= \int_0^{l_e} [m(\cos^2\theta_I \ \sin\Lambda_s \ \sin\Lambda_a \ \cos\Lambda_a \ -\cos\theta_I \ \sin\Lambda_s \ \cos^2\Lambda_a \\
&\beta_p + \beta_d\cos\theta_I - \beta_s\sin\theta_I \ )\{\Phi_c\} \ pct \ + (\cos^2\theta_I \ \sin\Lambda_a \ \cos\Lambda_s \ \cos\Lambda_a \\
&\{m\eta_m\cos(\theta_G+\phi_k) - m\zeta_m\sin(\theta_G+\phi_k)\} \ \{\Phi_c'\} \ \{\Phi_c'\}^T \\
&-(\cos^2\theta_I \ \cos^2\Lambda_s \ \cos^2\Lambda_a \ )(\{I_{\eta\eta} - I_{\zeta\zeta}\}\sin(\theta_G+\phi_k)\cos(\theta_G+\phi_k) \\
&-I_{\eta\zeta}\{\cos^2(\theta_G+\phi_k) - \sin^2(\theta_G+\phi_k)\} \ )\{\Phi_c'\} \ \{\Phi_c'\}^T \ ]dx
\end{aligned}
$$

$$
[K_{13}] = 0
$$

$$
\begin{aligned}
[K_{14}] &= \int_0^{l_e} [m(\cos^2\theta_I \ \sin\Lambda_s \ \cos\Lambda_s \ \sin^2\Lambda_a \ )\{\Phi_c\} \ \{\Phi_q\}^T \\
&+(\cos^2\theta_I \ \sin^2\Lambda_s \ \cos\Lambda_a \ -\cos^2\theta_I \ \cos^2\Lambda_s \ \cos^2\Lambda_a \ ) \\
&\{m\eta_m\cos(\theta_G+\phi_k) - m\zeta_m\sin(\theta_G+\phi_k)\} \ \{\Phi_c'\} \ \{\Phi_q\}^T \ ]dx
\end{aligned}
$$

$$
[K_{21}] = [K_{12}]^T
$$

$$
\begin{aligned}
[K_{22}] &= \int_0^{l_e} [-m(\cos^2\theta_I \ \sin^2\Lambda_a \ \sin^2\Lambda_s \ +\cos^2\theta_I \ \sin\Lambda_a \ \sin\Lambda_s \ )\{\Phi_c\} \ \{\Phi_c\}^T \\
&+(\cos^2\theta_I \ \sin^2\Lambda_s \ \cos^2\Lambda_a \ +\cos^2\theta_I
\end{aligned}
$$

$$\cos^2\Lambda_a\ \cos^2\Lambda_s\ )(\{Im_{\zeta\zeta}\sin^2(\theta_G+\phi_k)+Im_{\eta\eta}\cos^2(\theta_G+\phi_k)\}$$
$$+\{2I_{\eta\zeta}\cos(\theta_G+\phi_k)\sin(\theta_G+\phi_k)\}\ )\{\Phi_c'\}\ \{\Phi_c'\}^T\ ]dx$$

$$[K_{23}] = 0$$

$$[K_{24}] = \int_0^{l_e} [m(\cos^2\theta_I\ \cos\Lambda_s\ \cos\Lambda_a\ \sin\Lambda_a\ -\cos\theta_I\ \cos\Lambda_s$$
$$\cos^2\Lambda_a\ \beta_p+\beta_d\cos\theta_I-\beta_s\sin\theta_I\ )\{\Phi_c\}\ \{\Phi_q\}^T\ +(\cos^2\theta_I\ \sin^2\Lambda_s\ \cos\Lambda_a$$
$$-\cos^2\theta_I\ \cos^2\Lambda_s\ \cos^2\Lambda_a\ -\cos\theta_I\ \cos^2\Lambda_s\ \sin\Lambda_a\ \cos\Lambda_a\ \beta_p+\beta_d\cos\theta_I-\beta_s\sin\theta_I\ )$$
$$\{m\eta_m\sin(\theta_G+\phi_k)+m\zeta_m\cos(\theta_G+\phi_k)\}\ \{\Phi_c'\}\ \{\Phi_q\}^T\ ]dx$$

$$[K_{31}] = \int_0^{l_e} [\{m\eta_m\sin(\theta_G+\phi_k)+m\zeta_m\cos(\theta_G+\phi_k)\}\ (\cos^2\theta_I\ \cos^2\Lambda_s\ +a\cos\theta_I$$
$$+(e_1+e_2)\{\beta_s\cos\theta_I\ +\beta_d\sin\theta_I\ \}-e_1\beta_p\sin\theta_I$$
$$+\cos\theta_I\ \cos\Lambda_s\ \cos\Lambda_a\ \{-a(\beta_s\cos\theta_I\ +\beta_d\sin\theta_I\ )$$
$$+e_1+e_2+\sum_{i=1}^{n-1} l_{ei}+x_k\})\{\Phi_q\}\ \{\Phi_c\}^T$$
$$-[(2\cos^2\theta_I\ \sin\Lambda_s\ \cos\Lambda_s\ \cos\Lambda_a\ )(\{I_{\eta\eta}-I_{\zeta\zeta}\}\sin(\theta_G+\phi_k)\cos(\theta_G+\phi_k)$$
$$+I_{\eta\zeta}\{\cos^2(\theta_G+\phi_k)-\sin^2(\theta_G+\phi_k)\}\ )-(\{Im_{\zeta\zeta}\cos^2(\theta_G+\phi_k)+Im_{\eta\eta}\sin^2(\theta_G+\phi_k)\}$$
$$+\{2I_{\eta\zeta}\cos(\theta_G+\phi_k)\sin(\theta_G+\phi_k)\}\ )(\cos^2\theta_I\ \cos^2\Lambda_s\ \sin\Lambda_a\ \cos\Lambda_a$$
$$-\cos^2\theta_I\ \cos^2\Lambda_a\ \cos\Lambda_s\ \beta_p+\beta_d\cos\theta_I-\beta_s\sin\theta_I\ )]\{\Phi_q\}\ \{\Phi_c'\}^T\ ]dx$$

$$[K_{32}] = \int_0^{l_e} [(\cos^2\theta_I\ \sin\Lambda_s\ \sin\Lambda_a\ \{m\eta_m\sin(\theta_G+\phi_k)+m\zeta_m\cos(\theta_G+\phi_k)\}$$
$$+\cos^2\theta_I\ \sin\Lambda_s\ \cos\Lambda_s\ \sin\Lambda_a\ \{m\eta_m\cos(\theta_G+\phi_k)-m\zeta_m\sin(\theta_G+\phi_k)\}\ )\{\Phi_q\}\ \{\Phi_c\}^T$$
$$-[(\cos\theta_I\ \sin\Lambda_s\ \cos^2\Lambda_s\ \cos\Lambda_a\ )(\{I_{\eta\eta}-I_{\zeta\zeta}\}\sin(\theta_G+\phi_k)\cos(\theta_G+\phi_k)$$
$$+I_{\eta\zeta}\{\cos^2(\theta_G+\phi_k)-\sin^2(\theta_G+\phi_k)\}\ )+(\{Im_{\zeta\zeta}\cos^2(\theta_G+\phi_k)+Im_{\eta\eta}\sin^2(\theta_G+\phi_k)\}$$
$$+\{2I_{\eta\zeta}\cos(\theta_G+\phi_k)\sin(\theta_G+\phi_k)\}\ )(\cos^2\theta_I\ \cos\Lambda_s\ \sin\Lambda_s\ \cos\Lambda_a$$

$$- \cos^2\theta_I \ \cos^2\Lambda_a \ \cos\Lambda_s \ \beta_p + \beta_d \cos\theta_I - \beta_s \sin\theta_I \ )]\{\Phi_q\} \ \{\Phi'_c\}^T \ ]dx$$

$$\begin{aligned}
[K_{33}] \ = \ & \int_0^{l_e} [(ctit \ \cos^2\Lambda_a \ - \cos^2\theta_I \ \sin^2\Lambda_s \ \sin^2\Lambda_a \ ) \\
& (\{Im_{\zeta\zeta}\cos^2(\theta_G + \phi_k) + Im_{\eta\eta}\sin^2(\theta_G + \phi_k)\} \\
& -\{Im_{\zeta\zeta}\sin^2(\theta_G + \phi_k) + Im_{\eta\eta}\cos^2(\theta_G + \phi_k)\} \ )\{\Phi_q\} \ \{\Phi_q\}^T \ ]dx
\end{aligned}$$

$$\begin{aligned}
[K_{34}] \ = \ & \int_0^{l_e} [(\cos^2\theta_I \ \sin\Lambda_s \ \cos\Lambda_s \ + \cos^2\theta_I \ \cos\Lambda_s \ \cos\Lambda_a \ \sin\Lambda_s \ ) \\
& \{m\eta_m \sin(\theta_G + \phi_k) + m\zeta_m \cos(\theta_G + \phi_k)\} \ + (\cos^2\theta_I \ \cos^2\Lambda_s \ \cos\Lambda_a \ \sin\Lambda_a \ ) \\
& \{m\eta_m \cos(\theta_G + \phi_k) - m\zeta_m \sin(\theta_G + \phi_k)\} \ ]\{\Phi_q\} \ \{\Phi_q\}^T \ dx
\end{aligned}$$

$$\begin{aligned}
[K_{41}] \ = \ & \int_0^{l_e} [m(\cos^2\theta_I \ \sin\Lambda_s \ \cos\Lambda_s \ \sin^2\Lambda_s \ )\{\Phi_q\} \ \{\Phi_c\}^T \\
& +(\cos^2\theta_I \ \sin^2\Lambda_s \ \cos\Lambda_a \ - \cos^2\theta_I \ \cos^2\Lambda_s \ \cos^2\Lambda_a \ ) \\
& \{m\eta_m \cos(\theta_G + \phi_k) - m\zeta_m \sin(\theta_G + \phi_k)\} \ \{\Phi_q\} \ \{\Phi'_c\}^T \ ]dx
\end{aligned}$$

$$\begin{aligned}
[K_{42}] \ = \ & \int_0^{l_e} [m(\cos^2\theta_I \ \cos\Lambda_s \ \cos\Lambda_a \ \sin\Lambda_a \ )\{\Phi_q\} \ \{\Phi_c\}^T \ + (\cos^2\theta_I \ \sin^2\Lambda_s \ \cos\Lambda_a \\
& - \cos^2\theta_I \ \cos^2\Lambda_s \ \cos^2\Lambda_a \ )\{m\eta_m \sin(\theta_G + \phi_k) + m\zeta_m \cos(\theta_G + \phi_k)\} \ \{\Phi_q\} \ \{\Phi'_c\}^T \ ]dx
\end{aligned}$$

$$[K_{43}] \ = \ 0$$

$$[K_{44}] \ = \ \int_0^{l_e} [-m(\cos^2\theta_I \ \cos^2\Lambda_a \ + \cos^2\theta_I \ \sin^2\Lambda_s \ \sin^2\Lambda_a \ )\{\Phi_q\} \ \{\Phi_q\}^T \ ]dx$$

6.2.4 Vector $[V^L]_{14\times1}$

$$\left[V_{11}^L\right] = \int_0^{l_e} [-\{ m\eta_m \sin(\theta_G+\phi_k) + m\zeta_m \cos(\theta_G+\phi_k) \}\cos^2\theta_I \cos^2\Lambda_s \cos^2\Lambda_a \{\Phi_c\}$$

$$\{ma(2\cos\theta_I - \sin\theta_I)\cos\Lambda_a \cos\theta_I - m(e_1+e_2)\cos^2\Lambda_s \cos^2\theta_I \}\{\Phi_c\}$$

$$-\{m\eta_m \sin(\theta_G+\phi_k) + m\zeta_m \cos(\theta_G+\phi_k) \}\cos\Lambda_s \{\Phi_c\}$$

$$+\{m\eta_m \cos(\theta_G+\phi_k) - m\zeta_m \sin(\theta_G+\phi_k) \}\cos^2\Lambda_s \cos\Lambda_a (\ )\{\Phi_c'\}]dx$$

$$\left[V_{21}^L\right] = \int_0^{l_e} [\{-m(\ )\cos^2\Lambda_s \beta_p + \beta_d\cos\theta_I - \beta_s\sin\theta_I \}\{\Phi_c\}$$

$$\{ma(2\cos\theta_I - \sin\theta_I)\sin\theta_I \cos\Lambda_a \{\Phi_c\} + \{m\eta_m \sin(\theta_G+\phi_k)$$

$$+m\zeta_m \cos(\theta_G+\phi_k) \}(\ )\cos\theta_I \cos\Lambda_s \{\Phi_c'\}]dx$$

$$\left[V_{31}^L\right] = \int_0^{l_e} [\{(I_{\zeta\zeta} - I_{\eta\eta})\sin(\theta_G+\phi_k) \cos(\theta_G+\phi_k) \cos^2\theta_I \cos^2\Lambda_s \cos^2\Lambda_a$$

$$-I_{\eta\zeta} \{\sin^2(\theta_G+\phi_k) - \cos^2(\theta_G+\phi_k) \}\cos^2\theta_I \cos\Lambda_s \cos\Lambda_a + \{I_{\zeta\zeta} \sin(\theta_G+\phi_k)$$

$$+I_{\eta\eta} \cos(\theta_G+\phi_k) \}\sin\theta_I \cos\Lambda_a]\{\Phi_q\}\ dx$$

$$\left[V_{41}^L\right] = \int_0^{l_e} [-m\{ + (\ )$$

$$-a(\ \}\cos\theta_I \cos\Lambda_s \cos\Lambda_a \{\Phi_q\}]dx$$

6.2.5 Matrix $[M^1]_{14\times3}$

$$\left[M^1_{11}\right] = \int_0^{l_e} m[-\cos\Lambda_s \ \ \cos\theta_I \ + \sin\Lambda_s \ \cos\Lambda_a$$
$$+\sin\Lambda_s \ \sin\Lambda_a \ \ \sin\theta_I \ ]\{\Phi_c\} \ dx$$

$$\left[M^1_{12}\right] = \int_0^{l_e} m[\cos\Lambda_s \ \ \cos\theta_I \ - \sin\Lambda_s \ \cos\Lambda_a$$
$$-\sin\Lambda_s \ \sin\Lambda_a \ \ \sin\theta_I \ ]\{\Phi_c\} \ dx$$

$$\left[M^1_{13}\right] = \int_0^{l_e} m[\cos\Lambda_s \ \sin\theta_I \ + \sin\Lambda_s \ \sin\Lambda_a \ \cos\theta_I \ ]\{\Phi_c\} \ dx$$

$$\left[M^1_{21}\right] = \int_0^{l_e} m[\cos\Lambda_a \ \ \sin\theta_I \ - \sin\Lambda_s \ ]\{\Phi_c\} \ dx$$

$$\left[M^1_{22}\right] = \int_0^{l_e} m[-\cos\Lambda_a \ \ \sin\theta_I \ - \sin\Lambda_s \ ]\{\Phi_c\} \ dx$$

$$\left[M^1_{23}\right] = \int_0^{l_e} m[\cos\Lambda_a \ \cos\theta_I \ - \sin\Lambda_s \ \beta_p + \beta_d\cos\theta_I - \beta_s\sin\theta_I \ ]\{\Phi_c\} \ dx$$

$$\left[M^1_{31}\right] = 0$$

$$\left[M^1_{32}\right] = 0$$

$$\left[M^1_{33}\right] = 0$$

$$\left[M^1_{41}\right] = \int_0^{l_e} m[\cos\Lambda_s \ \cos\Lambda_a \ \ + \sin\Lambda_s \ \ \cos\theta_I$$
$$+\cos\Lambda_s \ \sin\Lambda_s \ \ \sin\theta_I \ ]\{\Phi_q\} \ dx$$

$$
\begin{aligned}
\left[M_{42}^{1}\right] &= \int_{0}^{l_e} m[\cos\Lambda_s \;\; \cos\Lambda_a \quad - \sin\Lambda_s \quad \cos\theta_I \\
&\quad - \cos\Lambda_s \;\; \sin\Lambda_s \quad \sin\theta_I \;\;]\{\Phi_q\}\; dx
\end{aligned}
$$

$$
\begin{aligned}
\left[M_{43}^{1}\right] &= \int_{0}^{l_e} m[\cos\Lambda_s \;\; \cos\Lambda_a \; \beta_p + \beta_d \cos\theta_I - \beta_s \sin\theta_I \;\; + \sin\theta_I \;\; \sin\Lambda_s \\
&\quad + \cos\Lambda_s \;\; \sin\Lambda_s \;\; \cos\theta_I \;\;]\{\Phi_q\}\; dx
\end{aligned}
$$

6.2.6 Matrix $[M^2]_{14\times 3}$

$$\begin{aligned}\left[M^2_{11}\right] &= \int_0^{l_e} m[-2\ \cos\Lambda_s\ \cos\theta_I\ -\ \cos\Lambda_s\ \cos\theta_I \\ &\quad +\ \cos^2\theta_I\ \sin\Lambda_s\ \cos\Lambda_a\ ]\{\Phi_c\}\ dx\end{aligned}$$

$$\begin{aligned}\left[M^2_{12}\right] &= \int_0^{l_e} m[-2\ \cos\Lambda_s\ \cos\theta_I\ +\ \cos\Lambda_s\ \cos\theta_I \\ &\quad -\ \cos^2\theta_I\ \sin\Lambda_s\ \cos\Lambda_a\ - \cos\Lambda_s\ \cos\theta_I\ ]\{\Phi_c\}\ dx\end{aligned}$$

$$\begin{aligned}\left[M^2_{13}\right] &= \int_0^{l_e} m[\ \cos\Lambda_s\ \cos\theta_I\ -\ \sin\Lambda_s\ \sin\Lambda_a\ \sin\theta_I\ + \cos\Lambda_s\ \cos\theta_I \\ &\quad + \sin\Lambda_a\ \cos\theta_I\ \cos\Lambda_a\ \sin^2\Lambda_s\ ]\{\Phi_c\}\ dx\end{aligned}$$

$$\begin{aligned}\left[M^2_{21}\right] &= \int_0^{l_e} m[-\ \cos\Lambda_s\ +\ \cos\theta_I\ \cos\Lambda_a\ \beta_p + \beta_d\cos\theta_I - \beta_s\sin\theta_I \\ &\quad +\ \cos^2\theta_I\ \sin\Lambda_a\ + \cos\Lambda_a\ \sin\theta_I\ ]\{\Phi_c\}\ dx\end{aligned}$$

$$\begin{aligned}\left[M^2_{22}\right] &= \int_0^{l_e} m[\ \cos\theta_I\ \cos\Lambda_a\ +\ \cos\Lambda_a \\ &\quad + \sin\Lambda_s\ - \sin\Lambda_s\ ]\{\Phi_c\}\ dx\end{aligned}$$

$$\begin{aligned}\left[M^2_{23}\right] &= \int_0^{l_e} m[\ \cos\theta_I\ \sin\Lambda_a \\ &\quad +\ \sin\Lambda_s\ ]\{\Phi_c\}\ dx\end{aligned}$$

$$\left[M^2_{31}\right] = 0$$

$$\left[M^2_{32}\right] = 0$$

$$\left[M^2_{33}\right] = 0$$

$$\left[M_{41}^{2}\right] = \int_{0}^{l_e} m[\cos\Lambda_s \ \cos\Lambda_a \quad \cos\theta_I \ + \sin\Lambda_s \quad \cos\theta_I$$
$$+ \cos\Lambda_s \ \sin\Lambda_s \quad \sin\theta_I \ ]\{\Phi_q\} \ dx$$

$$\left[M_{42}^{2}\right] = \int_{0}^{l_e} m[2 \ \cos\theta_I \ \cos\Lambda_s \ \cos\Lambda_a \ + \cos\Lambda_s \ \cos\Lambda_a \quad cti$$
$$+ \cos\Lambda_s \ \sin\Lambda_a \quad \sin\theta_I \ + \quad \sin\theta_I \ \sin\Lambda_s \ ]\{\Phi_q\} \ dx$$

$$\left[M_{43}^{2}\right] = \int_{0}^{l_e} -m[\ \sin\theta_I \ \cos\Lambda_s \ \sin\Lambda_a \ + \ \sin\Lambda_s \ \cos\theta_I \ + \ \cos\Lambda_s \ \cos\Lambda_a$$
$$+ \ \sin\theta_I \ \cos\Lambda_s \ \sin\Lambda_s \ ]\{\Phi_q\} \ dx$$

6.2.7 Matrix $[M^3]_{14\times 3}$

$$\begin{aligned}
[M^3_{11}] &= \int_0^{l_e} m[\cos\Lambda_s\ \{-\cos\psi_k\ (-a\sin\theta_I\ +(x_k+u_k)\ \cos\Lambda_s\ \sin\Lambda_a \\
&\quad +v_k\ \sin\Lambda_s\ \sin\Lambda_a\ +w_k\ \cos\Lambda_a\ )\} \\
&\quad +\sin\Lambda_s\ \cos\Lambda_a\ \{-\sin\psi_k\ \cos\theta_I\ (-a\sin\theta_I\ +(x_k+u_k)\ \cos\Lambda_s\ \sin\Lambda_a \\
&\quad +v_k\ \sin\Lambda_s\ \sin\Lambda_a\ +w_k\ \cos\Lambda_a\ ) \\
&\quad -\sin\psi_k\ \sin\theta_I\ (a\cos\theta_I\ -(x_k+u_k)\ \sin\Lambda_s\ +v_k\ \cos\Lambda_s\ )\} \\
&\quad +\sin\Lambda_s\ \sin\Lambda_a\ \{\cos\psi_k\ (a\cos\theta_I\ -(x_k+u_k)\ \sin\Lambda_s\ +v_k\ \cos\Lambda_s\ ) \\
&\quad +\sin\psi_k\ \cos\theta_I\ (e_1+e_2+\sum_{i=1}^{n-1}(l_e)_i+(x_k+u_k)\cos\Lambda_s\cos\Lambda_a\ +u_k\ \sin\Lambda_s \\
&\quad \cos\Lambda_a\ -w_k\ \sin\Lambda_a\ )\}]\{\Phi_c\}\ dx
\end{aligned}$$

$$\begin{aligned}
[M^3_{12}] &= \int_0^{l_e} m[\cos\Lambda_s\ \{-\sin\psi_k\ (-a\sin\theta_I\ +(x_k+u_k)\ \cos\Lambda_s\ \sin\Lambda_a \\
&\quad +v_k\ \sin\Lambda_s\ \sin\Lambda_a\ +w_k\ \cos\Lambda_a\ )\} \\
&\quad +\sin\Lambda_s\ \cos\Lambda_a\ \{\cos\psi_k\ \cos\theta_I\ (-a\sin\theta_I\ +(x_k+u_k)\ \cos\Lambda_s\ \sin\Lambda_a \\
&\quad +v_k\ \sin\Lambda_s\ \sin\Lambda_a\ +w_k\ \cos\Lambda_a\ )-\cos\psi_k \\
&\quad \sin\theta_I\ (a\cos\theta_I\ -(x_k+u_k)\ \sin\Lambda_s\ +v_k\ \cos\Lambda_s\ )\} \\
&\quad +\sin\Lambda_s\ \sin\Lambda_a\ \{\sin\psi_k\ (a\cos\theta_I\ -(x_k+u_k)\ \sin\Lambda_s\ +v_k\ \cos\Lambda_s\ ) \\
&\quad -\cos\psi_k\ \cos\theta_I\ (e_1+e_2+\sum_{i=1}^{n-1}(l_e)_i+(x_k+u_k)\cos\Lambda_s\cos\Lambda_a\ +v_k\ \sin\Lambda_s \\
&\quad \cos\Lambda_a\ -w_k\ \sin\Lambda_a\ )\}]\{\Phi_c\}\ dx
\end{aligned}$$

$$\begin{aligned}
[M^3_{13}] &= \int_0^{l_e} m[\cos\Lambda_s\ \cos\theta_I\ (e_1+e_2+\sum_{i=1}^{n-1}(l_e)_i+(x_k+u_k)\cos\Lambda_s\cos\Lambda_a\ +v_k\ \sin\Lambda_s\ \cos\Lambda_a \\
&\quad -w_k\ \sin\Lambda_a\ )-\sin\Lambda_s\ \cos\Lambda_a\ \cos\theta_I\ (a\ cti \\
&\quad -(x_k+u_k)\ \sin\Lambda_s\ +v_k\ \cos\Lambda_s\ )+\sin\Lambda_s\ \sin\Lambda_a\ \sin\theta_I \\
&\quad (e_1+e_2+\sum_{i=1}^{n-1}(l_e)_i+(x_k+u_k)\cos\Lambda_s\cos\Lambda_a\ +v_k\ \sin\Lambda_s\ \cos\Lambda_a
\end{aligned}$$

$$-w_k \ \sin\Lambda_a \ )]\{\Phi_c\} \ dx$$

$$\left[M_{21}^3\right] \ = \ \int_0^{l_e} m[\cos\Lambda_a \ \{\cos\psi_k \ (a\cos\theta_I \ - (x_k+u_k) \ \sin\Lambda_s \ + v_k \ \cos\Lambda_s \ )$$

$$+\sin\psi_k \ \cos\theta_I \ (e_1+e_2+\sum_{i=1}^{n-1}(l_e)_i + (x_k+u_k)\cos\Lambda_s\cos\Lambda_a \ + v_k \ \sin\Lambda_s \ \cos\Lambda_a$$

$$-w_k \ \sin\Lambda_a \ ) + \sin\Lambda_s \ \{\cos\theta_I \ \sin\psi_k \ (-a\sin\theta_I$$

$$+(x_k+u_k) \ \cos\Lambda_s \ \sin\Lambda_a \ + v_k \ \sin\Lambda_s \ \sin\Lambda_a \ + w_k \ \cos\Lambda_a \ )$$

$$+\sin\psi_k \ \sin\theta_I \ (a\cos\theta_I \ - (x_k+u_k) \ \sin\Lambda_s \ + v_k \ \cos\Lambda_s \ )\}]\{\Phi_c\} \ dx$$

$$\left[M_{22}^3\right] \ = \ \int_0^{l_e} m[\cos\Lambda_a \ \{\sin\psi_k \ (a\cos\theta_I \ - (x_k+u_k) \ \sin\Lambda_s$$

$$+v_k \ \cos\Lambda_s \ ) - \cos\psi_k \ \cos\theta_I \ (e_1+e_2+\sum_{i=1}^{n-1}(l_e)_i + (x_k+u_k)\cos\Lambda_s\cos\Lambda_a$$

$$+v_k \ \sin\Lambda_s \ \cos\Lambda_a \ - w_k \ \sin\Lambda_a \ )\} -$$

$$\sin\Lambda_s \ \{\cos\psi_k \ \cos\theta_I \ (-a\sin\theta_I \ + (x_k+u_k) \ \cos\Lambda_s \ \sin\Lambda_a$$

$$+v_k \ \sin\Lambda_s \ \sin\Lambda_a \ + w_k \ \cos\Lambda_a \ )$$

$$+\cos\psi_k \ \sin\theta_I \ (a\cos\theta_I \ - (x_k+u_k) \ \sin\Lambda_s \ + v_k \ \cos\Lambda_s \ )\}]\{\Phi_c\} \ dx$$

$$\left[M_{23}^3\right] \ = \ \int_0^{l_e} m[\cos\Lambda_a \ \{\sin\theta_I \ (e_1+e_2+\sum_{i=1}^{n-1}(l_e)_i + (x_k+u_k)\cos\Lambda_s\cos\Lambda_a \ + v_k \ \sin\Lambda_s \ \cos$$

$$-w_k \ \sin\Lambda_a \ )\} + \sin\Lambda_s \ \{\cos\theta_I \ (-a\sin\theta_I \ + (x_k+u_k) \ \cos\Lambda_s \ \sin\Lambda_a$$

$$+v_k \ \sin\Lambda_s \ \sin\Lambda_a \ + w_k \ \cos\Lambda_a \ ) + \cos\theta_I \ (a\cos\theta_I \ - (x_k+u_k) \ \sin\Lambda_s$$

$$+v_k \ \cos\Lambda_s \ )\}]\{\Phi_c\} \ dx$$

$$\left[M_{31}^3\right] \ = \ 0$$

$$\left[M_{32}^3\right] \ = \ 0$$

$$\left[M_{33}^{3}\right] = 0$$

$$\begin{aligned}\left[M_{41}^{3}\right] = & \int_0^{l_e} m[\cos\Lambda_s \ \cos\Lambda_a \ \{-\sin\psi_k \ \cos\theta_I \ (-a\sin\theta_I \\ & +(x_k+u_k) \ \cos\Lambda_s \ \sin\Lambda_a \ + v_k \ \sin\Lambda_s \ \sin\Lambda_a \ + w_k \ \cos\Lambda_a \ ) \\ & -\sin\psi_k \ \sin\theta_I \ (a\cos\theta_I \ - (x_k+u_k) \ \sin\Lambda_s \ + v_k \ \cos\Lambda_s \ )\} \\ & +\sin\Lambda_s \ \{\cos\psi_k \ (-a\sin\theta_I \ + (x_k+u_k) \\ & \cos\Lambda_s \ \sin\Lambda_a \ + v_k \ \sin\Lambda_s \ \sin\Lambda_a \ + w_k \ \cos\Lambda_a \ )\} \\ & +\cos\Lambda_s \ \sin\Lambda_a \ \{\cos\psi_k \ (a\cos\theta_I \ - (x_k+u_k) \ \sin\Lambda_s \ + v_k \ \cos\Lambda_s \ ) \\ & +\sin\psi_k \ \cos\theta_I \ (e_1+e_2+\sum_{i=1}^{n-1}(l_e)_i + (x_k+u_k)\cos\Lambda_s\cos\Lambda_a \ + v_k \\ & \sin\Lambda_s \ \cos\Lambda_a \ - w_k \ \sin\Lambda_a \ )\}]\{\Phi_q\} \ dx\end{aligned}$$

$$\begin{aligned}\left[M_{42}^{3}\right] = & \int_0^{l_e} m[\cos\Lambda_s \ \cos\Lambda_a \ \{\cos\psi_k \ \cos\theta_I \ (-a\sin\theta_I \\ & +(x_k+u_k) \ \cos\Lambda_s \ \sin\Lambda_a \ + v_k \ \sin\Lambda_s \ \sin\Lambda_a \ + w_k \ \cos\Lambda_a \ ) \\ & +\sin\theta_I \ \cos\psi_k \ (a\cos\theta_I \ - (x_k+u_k) \ \sin\Lambda_s \ + v_k \ \cos\Lambda_s \ )\} \\ & +\sin\Lambda_s \ \{\sin\psi_k \ (-a\sin\theta_I \ + (x_k+u_k) \ \cos\Lambda_s \ \sin\Lambda_a \\ & +v_k \ \sin\Lambda_s \ \sin\Lambda_a \ + w_k \ \cos\Lambda_a \ )\} \\ & +\cos\Lambda_s \ \sin\Lambda_a \ \{\sin\psi_k \ (a\cos\theta_I \ - (x_k+u_k) \ \sin\Lambda_s \ + v_k \ \cos\Lambda_s \ ) \\ & -\cos\psi_k \ \cos\theta_I \ (e_1+e_2+\sum_{i=1}^{n-1}(l_e)_i + (x_k+u_k)\cos\Lambda_s\cos\Lambda_a \ + v_k \ \sin\Lambda_s \\ & \cos\Lambda_a \ - w_k \ \sin\Lambda_a \ )\}]\{\Phi_q\} \ dx\end{aligned}$$

$$\begin{aligned}\left[M_{43}^{3}\right] = & \int_0^{l_e} m[\cos\Lambda_s \ \cos\Lambda_a \ \{-\cos\theta_I \ (-a\sin\theta_I \\ & +(x_k+u_k) \ \cos\Lambda_s \ \sin\Lambda_a \ + v_k \ \sin\Lambda_s \ \sin\Lambda_a \ + w_k \ \cos\Lambda_a \ ) \\ & -\sin\theta_I \ (a\cos\theta_I \ - (x_k+u_k) \ \sin\Lambda_s \ + v_k \ \cos\Lambda_s \ )\} \\ & -\sin\Lambda_s \ \{\cos\theta_I \ (e_1+e_2+\sum_{i=1}^{n-1}(l_e)_i + (x_k+u_k)\cos\Lambda_s\cos\Lambda_a\end{aligned}$$

$$+v_k \ \sin\Lambda_s \ \cos\Lambda_a \ - w_k \ \sin\Lambda_a \ )\} + \cos\Lambda_s \ \sin\Lambda_s$$

$$\{\sin\theta_I \ (e_1 + e_2 + \sum_{i=1}^{n-1}(l_e)_i + (x_k + u_k)\cos\Lambda_s\cos\Lambda_a \ + v_k \ \sin\Lambda_s \ \cos\Lambda_a$$

$$-w_k \ \sin\Lambda_a \ )\}]\{\Phi_q\} \ dx$$

6.2.8 Matrix $[M^4]_{14\times 3}$

$$
\begin{aligned}
\left[M_{11}^4\right] &= \int_0^{l_e} [-m\cos\Lambda_s\ \cos\Lambda_a\ \{a\cos\theta_I \\
&\quad -(\beta_s\cos\theta_I - \beta_d\sin\theta_I + 1)\ (\sum_{i=1}^{n-1}(l_e)_i + x_k\ ) - (e_1+e_2)\}\cos\psi_k \\
&\quad +m\cos\Lambda_s\ \cos\Lambda_a\ \{(\sum_{i=1}^{n-1}(l_e)_i + x_k\ )\sin\psi_k\ \}]\{\Phi_c\}\ dx
\end{aligned}
$$

$$
\begin{aligned}
\left[M_{12}^4\right] &= \int_0^{l_e} [-m\ \cos\Lambda_s\ \cos\Lambda_a\ a\cos\theta_I \\
&\quad -(\beta_s\cos\theta_I - \beta_d\sin\theta_I + 1)\ (\sum_{i=1}^{n-1}(l_e)_i + x_k\ ) - (e_1+e_2)\}\sin\psi_k \\
&\quad -m\cos\Lambda_s\ \cos\Lambda_a\ \{(\sum_{i=1}^{n-1}(l_e)_i + x_k\ )\cos\psi_k\ \}]\{\Phi_c\}\ dx
\end{aligned}
$$

$$
\left[M_{13}^4\right] = \int_0^{l_e} [ma\cos\Lambda_s\ \cos\theta_I\ (2\cos\theta_I\ - \sin\theta_I\ )\{\Phi_c\}\ \{\Phi_c\}\ ]dx
$$

$$
\left[M_{21}^4\right] = \int_0^{l_e} [-m\cos\Lambda_a\ \cos\Lambda_s\ \cos\theta_I\ \{2a\sin\psi_k\ \cos\theta_I\ + (\sum_{i=1}^{n-1}(l_e)_i + x_k\ )\cos\psi_k\ \}\{\Phi_c\}\ ]dx
$$

$$
\left[M_{22}^4\right] = \int_0^{l_e} [m\sin\Lambda_a\ \cos\Lambda_a\ \cos\Lambda_s\ \{2a\cos\psi_k\ \cos\theta_I\ - (\sum_{i=1}^{n-1}(l_e)_i + x_k\ )\sin\psi_k\ \}\{\Phi_c\}\ ]dx
$$

$$
\left[M_{23}^4\right] = \int_0^{l_e} [\{-m\cos\Lambda_a\ \cos^2\Lambda_s\ (\sum_{i=1}^{n-1}(l_e)_i + x_k\ )(\beta_p + \beta_d\cos\theta_I - \beta_s\sin\theta_I\ - \dot{\theta}_I\ )\}\{\Phi_c\}\ dx
$$

$$
\left[M_{31}^4\right] = 0
$$

$$
\left[M_{32}^4\right] = 0
$$

$$
\left[M_{33}^4\right] = 0
$$

$$\left[M_{41}^{4}\right] = [-m\ cls\ \cos\Lambda_a\ \cos\theta_I\ \{a\cos\theta_I\ - (\beta_s\cos\theta_I - \beta_d\sin\theta_I + 1)\ (\sum_{i=1}^{n-1}(l_e)_i + x_k\ )$$
$$-(e_1 + e_2)\}\sin\psi_k\ \{\Phi_q\}\ ]dx$$

$$\left[M_{42}^{4}\right] = [m\ cls\ \cos\theta_I\ \{a\cos\theta_I\ - (\beta_s\cos\theta_I - \beta_d\sin\theta_I + 1)\ (\sum_{i=1}^{n-1}(l_e)_i + x_k\ )$$
$$-(e_1 + e_2)\}\cos\psi_k\ \{\Phi_q\}\ ]dx$$

$$\left[M_{43}^{4}\right] = m[\{(e_1 + e_2) + 2(\sum_{i=1}^{n-1}(l_e)_i + x_k\ )\}\cos\theta_I\ \cos\Lambda_s\ \cos^2\Lambda_a\ \{\Phi_q\}\ ]dx$$

6.2.10 Nonlinear Vector $[V^{NL}]_{14\times 1}$

$$\begin{aligned}
\left[V_{11}^{NL}\right] &= \int_0^{l_e} -m[2\dot{\theta}_I \cos\Lambda_s \cos\Lambda_a \cos\theta_I \{\Phi_c\}\{\Phi_c\}^T\{V\} \\
&+(\dot{\theta}_I^{\,2} \cos\theta_I \cos\Lambda_s + \dot{\theta}_I \cos\Lambda_s \dot{\theta}_z - \ddot{\theta}_I)\{\Phi_c\}\{\Phi_c\}^T\{W\} \\
&+\ddot{\theta}_I \cos^2\Lambda_a \{\Phi_c\}\{\Phi_q\}^T\{U\} + 2\dot{\theta}_I \cos\Lambda_a \{\Phi_c\}\{\Phi_q\}^T\{\dot{U}\}]dx
\end{aligned}$$

$$\begin{aligned}
\left[V_{21}^{NL}\right] &= \int_0^{l_e} -m[(\dot{\theta}_I - \ddot{\theta}_I)\cos\Lambda_s \{\Phi_c\}\{\Phi_c\}^T\{V\} \\
&-\dot{\theta}_I \cos\Lambda_s \cos\Lambda_a \{\Phi_c\}\{\Phi_c\}^T\{\dot{V}\} + \dot{\theta}_I^{\,2} \cos^2\Lambda_a \cos\Lambda_s \{\Phi_c\}\{\Phi_c\}^T\{W\}]dx
\end{aligned}$$

$$\left[V_{31}^{NL}\right] = 0$$

$$\begin{aligned}
\left[V_{41}^{NL}\right] &= \int_0^{l_e} m[(-\dot{\theta}_I)\cos\Lambda_s \cos\Lambda_a \{\Phi_q\}\{\Phi_c\}^T\{\dot{V}\} \\
&-(\dot{\theta}_I + \dot{\theta}_I^{\,2})\cos\theta_I \cos\Lambda_s \cos\Lambda_a \{\Phi_q\}\{\Phi_c\}^T\{W\} \\
&-\dot{\theta}_I \cos\Lambda_s \{\Phi_q\}\{\Phi_q\}^T\{U\}]dx
\end{aligned}$$

6.3 ELEMENT MATRICES ASSOCIATED WITH STRAIN ENERGY VARIATION

The element matrices associated with the strain energy variation are derived by substituting the assumed expressions for the displacement functions (Eq. 6.2), in the strain energy variation, δU_i (Eq. 5.9), and carrying out the integration over the length of the beam element. The resulting variation of the strain energy can be written in the form;

$$\delta U_i = E_o l^3 \{\delta q\}^T \{[K^E]\{q\} + \{F^E\}\} \tag{6.8}$$

Where

$\{q\}$ represents the vector of unknown degree of freedom.

$[K^E]$ is the linear stiffness matrix. $\{F^E\}$ is the nonlinear stiffness vector.

The detailed expressions for the $[K^E]$ and $\{F^E\}$ are as follows:

6.3.1 Linear Stiffness Matrix $[K^E]$

The linear stiffness matrix $[K^E]$ is given by the following sub-matrices:

$$[K_{11}^E] = \int_0^{l_e} \left[(\overline{EI_{\zeta\zeta}} \cos\theta_G - \overline{EI_{\eta\zeta}} \sin\theta_G)\{\Phi_c''\}\{\Phi_c''\}^T\right] dx$$

$$[K_{12}^E] = \int_0^{l_e} \left[(\overline{EI_{\zeta\eta}} \cos\theta_G - \overline{EI_{\eta\eta}} \sin\theta_G)\{\Phi_c''\}\{\Phi_c''\}^T\right] dx$$

$$[K_{13}^E] = \int_0^{l_e} \left[(\overline{EAD_1})\{\Phi_c''\}\{\Phi_q''\}^T + (\tau\overline{EAD_1'})\{\Phi_c''\}\{\Phi_q''\}^T\right] dx$$

$$[K_{14}^E] = \int_0^{l_e} \left[(-\overline{EA\eta_a})\{\Phi_c''\}\{\Phi_q''\}^T\right] dx$$

$$\left[K^E_{21}\right] = \left[K^E_{12}\right]^T$$

$$\left[K^E_{22}\right] = \int_0^{l_e} \left[(\overline{EI_{\eta\eta}}\cos\theta_G - \overline{EI_{\zeta\eta}}\sin\theta_G)\{\Phi''_c\}\{\Phi''_c\}^T\right]dx$$

$$\left[K^E_{23}\right] = \int_0^{l_e} \left[(\overline{EAD_2})\{\Phi''_c\}\{\Phi''_q\}^T + (\tau\overline{EAD'_2})\{\Phi''_c\}\{\Phi''_q\}^T\right]dx$$

$$\left[K^E_{24}\right] = \int_0^{l_e} \left[(-\overline{EA\zeta_a})\{\Phi''_c\}\{\Phi''_q\}^T\right]dx$$

$$\left[K^E_{31}\right] = [K^E_{13}]^T$$

$$\left[K^E_{32}\right] = [K^E_{23}]^T$$

$$\begin{aligned}\left[K^E_{33}\right] = &\int_0^{l_e} [(EAD_3)\{\Phi''_q\}\{\Phi''_q\}^T + (\tau EAD'_2)\{\Phi''_q\}\{\Phi''_q\}^T \\ &+\{\Phi'_q\}\{\Phi''_q\}^T + (\tau^2 EAD'_3 + GJ)\{\Phi'_q\}\{\Phi''_q\}^T]dx\end{aligned}$$

$$\left[K^E_{34}\right] = \int_0^{l_e} \left[(-EAD_o)\{\Phi''_q\}\{\Phi''_q\}^T + (-\tau EAD'_2)\{\Phi'_q\}\{\Phi''_q\}^T\right]dx$$

$$\left[K^E_{41}\right] = [K^E_{14}]^T$$

$$\left[K^E_{42}\right] = [K^E_{24}]^T$$

$$\left[K^E_{43}\right] = [K^E_{34}]^T$$

$$\left[K_{44}^{E}\right] = \int_{0}^{l_e} \left[(EA)\{\Phi_q'\}\,\{\Phi_q''\}^T\right] dx$$

6.3.2 Simplification of Linear Stiffness Matrix $[K^E]$

A simplified form of the linear stiffness matrix can be obtained by making the following assumptions:

1. The elastic axis is coincident with the modulus weighted centroid.
2. The warping function is considered only in torsion.

If the properties like EI, $EI_{\eta\eta}, EI_{\zeta\zeta}$ etc.. vary linearly along the element length, various quantities defined in the linear stiffness matrices can be expressed as

$$\overline{EI_{\zeta\zeta}} = \left[EI_{\zeta\zeta_1} + \left(\frac{EI_{\zeta\zeta_1} - EI_{\zeta\zeta_2}}{L}\right)x\right]\cos\theta_G - \left[EI_{\zeta\eta_1} + \left(\frac{EI_{\zeta\eta_1} - EI_{\zeta\eta_2}}{L}\right)x\right]\sin\theta_G$$

$$\overline{EI_{\eta\zeta}} = \left[EI_{\eta\zeta_1} + \left(\frac{EI_{\eta\zeta_1} - EI_{\eta\zeta_2}}{L}\right)x\right]\cos\theta_G - \left[EI_{\eta\eta_1} + \left(\frac{EI_{\eta\eta_1} - EI_{\eta\eta_2}}{L}\right)x\right]\sin\theta_G$$

$$\overline{EI_{\zeta\eta}} = \left[EI_{\zeta\zeta_1} + \left(\frac{EI_{\zeta\zeta_1} - EI_{\zeta\zeta_2}}{L}\right)x\right]\sin\theta_G + \left[EI_{\zeta\eta_1} + \left(\frac{EI_{\zeta\eta_1} - EI_{\zeta\eta_2}}{L}\right)x\right]\sin\theta_G$$

$$\overline{EI_{\eta\eta}} = \left[EI_{\eta\zeta_1} + \left(\frac{EI_{\eta\zeta_1} - EI_{\eta\zeta_2}}{L}\right)x\right]\sin\theta_G + \left[EI_{\eta\eta_1} + \left(\frac{EI_{\eta\eta_1} - EI_{\eta\eta_2}}{L}\right)x\right]\cos\theta_G$$

$$\overline{EAD_1} = \left[EAD_{11} + \left(\frac{EAD_{12} - EAD_{11}}{L}\right)x\right]\cos\theta_G - \left[EAD_{21} + \left(\frac{EAD_{22} - EAD_{21}}{L}\right)x\right]\sin\theta_G$$

$$\overline{EAD'_1} = \left[EAD'_{11} + \left(\frac{EAD'_{12} - EAD'_{11}}{L}\right)x\right]\cos\theta_G - \left[EAD'_{21} + \left(\frac{EAD'_{22} - EAD'_{21}}{L}\right)x\right]\sin\theta_G$$

$$\overline{EA\eta_a} = \left[EA\eta_{a1} + \left(\frac{EA\eta_{a2} - EA\eta_{a1}}{L}\right)x\right]\cos\theta_G - \left[EA\zeta_{a1} + \left(\frac{EA\zeta_{a2} - EA\zeta_{a1}}{L}\right)x\right]\sin\theta_G$$

$$EAD_3 = \left[EAD_{31} + \left(\frac{EAD_{32} - EAD_{31}}{L}\right) x\right]$$

$$EAD_5 = \left[EAD_{51} + \left(\frac{EAD_{52} - EAD_{51}}{L}\right) x\right]$$

$$EAD_3' = \left[EAD_{31}' + \left(\frac{EAD_{52}' - EAD_{51}'}{L}\right) x\right]$$

$$\begin{aligned} GJ &= \left[GJ_{o1} + \left(\frac{GJ_{o2} - GJ_{o1}}{L}\right) x\right] - 2\left[GJ_{11} + \left(\frac{GJ_{12} - GJ_{11}}{L}\right) x\right] \\ &\quad + \left[GJ_{21} + \left(\frac{GJ_{22} - GJ_{21}}{L}\right) x\right] \end{aligned}$$

$$EAD_o = \left[EAD_{o1} + \left(\frac{EAD_{o2} - EAD_{o1}}{L}\right) x\right]$$

$$EA = \left[EA_1 + \left(\frac{EA_2 - EA_1}{L}\right) x\right]$$

Where the subscript 1 and 2 refer to the properties at the two end nodes. Substituting these expressions in the stiffness matrix leads to several integrals along the length of the element, which are given as

$$\int_0^{l_e} \{\Phi_c''\} \{\Phi_c''\}^T \, dx = \frac{1}{l_e^3} \begin{bmatrix} 12 & 6l_e & -12 & 6l_e \\ 6l_e & 4l_e^2 & -6l_e & 2l_e^2 \\ -12 & -6l_e & 12 & -6l_e \\ 6l_e & 2l_e^2 & -6l_e & 4l_e^2 \end{bmatrix}$$

$$\int_0^{l_e} \{\Phi_q'\} \{\Phi_q''\}^T \, dx = \frac{1}{3l_e} \begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & 8 \\ 1 & 8 & 7 \end{bmatrix}$$

$$\int_0^{l_e} \{\Phi_c''\}\,\{\Phi_q''\}^T\,dx = \frac{1}{l_e^3}\begin{bmatrix} 0 & 0 & 0 \\ -4l_e & 8l_e & -4l_e \\ 0 & 0 & 0 \\ 4l_e & -8l_e & 4l_e \end{bmatrix}$$

$$\int_0^{l_e} \{\Phi_q''\}\,\{\Phi_q''\}^T\,dx = \frac{1}{l_e^3}\begin{bmatrix} 16 & -32 & 16 \\ -32 & 64 & -32 \\ 16 & -32 & 16 \end{bmatrix}$$

$$\int_0^{l_e} \{\Phi_q''\}\,\{\Phi_q''\}^T\,dx = \frac{1}{l_e^2}\begin{bmatrix} -4 & 0 & 4 \\ 8 & 0 & -8 \\ -4 & 0 & 4 \end{bmatrix}$$

$$\int_0^{l_e} \{\Phi_c''\}\,\{\Phi_q''\}^T\,dx = \frac{1}{l_e^2}\begin{bmatrix} 4 & -8 & 4 \\ 3l_e & -4l_e & l_e \\ -4 & 8 & -4 \\ l_e & -4l_e & 3l_e \end{bmatrix}$$

$$\int_0^{l_e} \{\Phi_q'\}\,\{\Phi_q''\}^T\,dx = \begin{bmatrix} -4 & 8 & -4 \\ 0 & 0 & 0 \\ 4 & -8 & 4 \end{bmatrix}$$

$$\int_0^{l_e} x\{\Phi_c''\}\,\{\Phi_c''\}^T\,dx = \frac{1}{l_e^2}\begin{bmatrix} 6 & 2l_e & -6 & 4l_e \\ 2l_e & 5l_e^2 & -2l_e & l_e^2 \\ -6 & -2l_e & 6 & -4l_e \\ 4l_e & l_e^2 & -4l_e & 3l_e^2 \end{bmatrix}$$

$$\int_0^{l_e} x\{\Phi_q'\}\,\{\Phi_q''\}^T\,dx = \frac{1}{6}\begin{bmatrix} 3 & -4 & 1 \\ -4 & 16 & -12 \\ 1 & -12 & 11 \end{bmatrix}$$

$$\int_0^{l_e} x\{\Phi_c''\}\,\{\Phi_q''\}^T\,dx = \frac{1}{l_e^2}\begin{bmatrix} 4 & -8 & 4 \\ 0 & 0 & 0 \\ -4 & 8 & -4 \\ 4l_e & -8l_e & 4l_e \end{bmatrix}$$

$$\int_0^{l_e} x\{\Phi_q''\}\,\{\Phi_q''\}^T\,dx = \frac{1}{l_e^2}\begin{bmatrix} 8 & -16 & 8 \\ -16 & 32 & -16 \\ 8 & -16 & 8 \end{bmatrix}$$

$$\int_0^{l_e} x\{\Phi_q''\}\,\{\Phi_q''\}^T\,dx = \frac{1}{3l_e}\begin{bmatrix} -2 & -8 & 10 \\ 4 & 16 & 20 \\ -2 & -8 & 10 \end{bmatrix}$$

$$\int_0^{l_e} x\{\Phi_c''\}\,\{\Phi_q''\}^T\,dx = \frac{1}{3l_e}\begin{bmatrix} 3 & -12 & 9 \\ 2l_e & -4l_e & 2l_e \\ -3 & 12 & -9 \\ 7l_e & -8l_e & 7l_e \end{bmatrix}$$

$$\int_0^{l_e} x\{\Phi_q'\}\,\{\Phi_q''\}^T\,dx = \frac{1}{3l_e}\begin{bmatrix} -2 & 4 & -2 \\ -8 & 16 & -8 \\ 10 & 20 & 10 \end{bmatrix}$$

(6.9)

6.3.3 Nonlinear Stiffness Vector $[F^E]$

The nonlinear stiffness vector $[F^E]$ is given by the following sub-vectors:

$$
\begin{aligned}
\{F_1^E\} = & \int_0^{l_e} \{[\underline{(\bar{V}_x)(v_x)} - (\bar{S}_x' \sin\theta_G\ )(v_{xx}\cos\theta_G\ + w_{xx}\sin\theta_G\ )] \\
& +[-(\overline{EI_{\zeta\eta}}\cos\theta_G \overline{EI_{\eta\eta}}\sin\theta_G\ )(\phi)(v_{xx}) + (\overline{EI_{\zeta\zeta}}\cos\theta_G\ - \overline{EI_{\eta\zeta}}\sin\theta_G\ )(\phi)(w_{xx}) \\
& -\frac{1}{2}(\overline{EA\eta_a}(v_x^2 + w_x^2) + (\bar{S}_x' \cos\theta_G\ )(-v_x\sin\theta_G\ + w_x\cos\theta_G\ ) \\
& +(\bar{M}_\eta' \cos\theta_G\ - \bar{M}_\zeta' \sin\theta_G\ )(\phi) - \frac{1}{2}(\overline{EAC_1}(\phi_x^2)]\{\Phi_c''\}\ \}dx
\end{aligned}
$$

$$
\begin{aligned}
\{F_2^E\} = & \int_0^{l_e} \{[\underline{(\bar{V}_x)(w_x)} - (\bar{S}_x' \cos\theta_G\ )(v_{xx}\cos\theta_G\ + w_{xx}\sin\theta_G\ )] \\
& +[-(\overline{EI_{\zeta\eta}}\sin\theta_G \overline{EI_{\eta\eta}}\cos\theta_G\ )(\phi)(v_{xx}) + (\overline{EI_{\zeta\zeta}}\sin\theta_G\ + \overline{EI_{\eta\zeta}}\cos\theta_G\ )(\phi)(w_{xx}) \\
& -\frac{1}{2}(\overline{EA\zeta_a}(v_x^2 + w_x^2) + (\bar{S}_x' \sin\theta_G\ )(-v_x\sin\theta_G\ + w_x\cos\theta_G\ ) \\
& +(\bar{M}_\eta' \sin\theta_G\ + \bar{M}_\zeta' \cos\theta_G\ )(\phi) - \frac{1}{2}(\overline{EAC_2})(\phi_x^2)]\{\Phi_c''\}\ \}dx
\end{aligned}
$$

$$
\begin{aligned}
\{F_3^E\} = & \int \{[(\bar{M}_\eta' \cos\theta_G\ - \bar{M}_\zeta' \sin\theta_G\ )(v_{xx}) + (\bar{M}_\eta' \sin\theta_G\ + \bar{M}_\zeta' \cos\theta_G\ )(w_{xx})\{\Phi_q\} \\
& [(\tau_o)[(\overline{EAD_2'}v_{xx} - \overline{EAD_1'}w_{xx})(\phi) \\
& +\frac{1}{2}EAD_o'(v_x^2 + w_x^2) + \frac{1}{2}EAD_4'(\phi_x^2)] \\
& -(GJ_o - GJ_1)(\phi_o) - \underline{(\bar{T}_x)(\phi_x)}]\{\Phi_q'\} \\
& -[\frac{1}{2}EAD_o(v_x^2 + w_x^2) + \frac{1}{2}EAD_4(\phi_x^2) + (\overline{EAD_2}v_{xx} - \overline{EAD_1}w_{xx}\phi]\{\Phi_q''\}\ \}dx
\end{aligned}
$$

$$
\{F_4^E\} = \int \{[(\overline{EA\zeta_z}v_{xx} - \overline{EA\eta_a})(\phi) + \frac{1}{2}EA(v_x^2 + w_x^2) + \frac{1}{2}EAC_o(\phi_x^2)]\{\Phi_q'\}\ \}dx
$$

The underlined terms in $\{F_1^E\}$, $\{F_2^E\}$ and $\{F_3^E\}$ are associated with the axial strain at the elastic axis $(u_x + \frac{1}{2}v_x^2 + \frac{1}{2}w_x^2)$ of the blade. These nonlinear terms are modified to a set of linear terms by substituting the axial stress by the axial inertial force.

The description of the substitution approach [19, 29] is given in next section for clarity.

6.3.4 Treatment of Nonlinear Terms Associated with Axial Strain at Elastic Axis

The equation of motion corresponding to the axial degree of freedom can be written symbolically as [19,29]:

$$\frac{E_o}{m\Omega^2}[-\bar{V}_x]_x - \bar{Z}_u = 0 \tag{6.10}$$

The axial stress resultant (Eq. 5.10) can be rewritten as:

$$\bar{V}_x = EA[u_x + \frac{1}{2}v_x^2 + \frac{1}{2}w_x^2] + \bar{F} \tag{6.11}$$

Where $\bar{F}$ contains all the additional higher order terms, which can be neglected. The simplified expressions for $\bar{V}_x$ can be written as:

$$\bar{V}_x \approx EA[u_x + \frac{1}{2}v_x^2 + \frac{1}{2}w_x^2] \tag{6.12}$$

Neglecting all the higher order terms and also the time derivative terms, the distributed inertia force $\bar{Z}_u$ can be written as:

$$\bar{Z}_u = C_1[\sum_{i=1}^{n-1}(l_e)_i + x_k] + C_2 \tag{6.13}$$

where

$$\begin{aligned} C_1 &= \{1 - (\beta_s \cos\theta_I + \beta_d \sin\theta_I)\} \cos\theta_I^2 \cos\Lambda_s \cos\Lambda_a \\ C_2 &= \{(e_1 + e_2) - a(\beta_d \sin\theta_I + \beta_s \cos\theta_I)\} \cos\theta_I^2 \cos\Lambda_s \cos\Lambda_a \end{aligned}$$

Integrating Eq. 6.10,

$$\frac{E_o}{m\Omega^2}\bar{V}_x = \int_{nl_e - x_k}^{l} \bar{Z}_u dx \tag{6.14}$$

$\bar{V}_x$ can be written as

$$\bar{V}_x = f(\xi) = a_1\xi^2 + a_2\xi + a_3 \tag{6.15}$$

where

$$\xi = \frac{x_k}{l_e}$$

a_1, a_2, a_3 are given as:

$$a_1 = \frac{m\Omega^2}{E_o}[-\frac{1}{2}C_1 l_e^2]$$

$$a_2 = \frac{m\Omega^2}{E_o}[-C_1\sum_{i=1}^{n-1}(l_e)_i - C_2]l_e$$

$$a_3 = \frac{m\Omega^2}{E_o}[\frac{1}{2}C_1\{\sum_{i=1}^{N}(l_e)_i + \sum_{i=1}^{n-1}(l_e)_i\} + C_2][\sum_{i=1}^{N}(l_e)_i - \sum_{i=1}^{n-1}(l_e)_i]$$

Combining Eqs. 6.15 with 6.12, the axial strain at the elastic axis can be written in terms of axial inertia force as

$$[u_x + \frac{1}{2}v_x^2 + \frac{1}{2}w_x^2] = \frac{f(\xi)}{EA} \tag{6.16}$$

Using Eq. 6.16, the underlined terms in $\{F_1^E\}, \{F_2^E\}$ and $\{F_3^3\}$ are modified to a set of linear stiffness matrices, which are given as:

$$\int_0^{l_e} \bar{V}_x v_x\{\Phi_c'\}dx = l_e\int_0^1 f(\xi)\{\Phi_c'\}\{\Phi_c'\}^T d\xi\{V\} = [K_{11}^{E'}]\{V\}$$

$$\int_0^{l_e} \bar{V}_x w_x \{\phi_c'\} dx = l_e \int_0^1 f(\xi)\{\Phi_c'\}\{\Phi_c'\}^T d\xi \{W\} = [K_{22}^{E'}]\{W\}$$

$$\int_0^{l_e} \bar{T}_x \phi_x \{\phi_q'\} dx = \left[\frac{EAC_o}{EA}\right] l_e \int_0^1 f(\xi)\{\Phi_q'\}\{\Phi_q'\}^T d\xi \{\Phi\} = [K_{33}^{E'}]\{\Phi\}$$

Combining these submatrices, a linear stiffness matrix $[K^{E'}]$ is obtained, which can be written as

$$\left[K^{E'}\right] = \begin{bmatrix} [K_{11}^{E'}] & [0] & [0] & [0] \\ [0] & [K_{22}^{E'}] & [0] & [0] \\ [0] & [0] & [K_{33}^{E'}] & [0] \\ [0] & [0] & [0] & [0] \end{bmatrix} \tag{6.17}$$

where

$$[K_{11}^{E'}] = [K_{22}^{E'}] = (a_1 + a_2 + a_3)[A_1] - (2a_1 + a_2)[A_2] + (2a_1)[A_3]$$

$$[K_{33}^{E'}] = (a_1 + a_2 + a_3)[B_1] - (2a_1 + a_2)[B_2] + (2a_1)[B_3]$$

where

$$[A_1] = \frac{1}{30l_e}\begin{bmatrix} 36 & 3l_e & -36 & 3l_e \\ 3l_e & 4l_e^2 & -3l_e & -l_e^2 \\ -36 & -3l_e & 36 & -3l_e \\ 3l_e & -l_e^2 & -3l_e & 4l_e^2 \end{bmatrix} \tag{6.18}$$

$$[A_2] = \frac{1}{60l_e}\begin{bmatrix} 36 & 0 & -36 & 6l_e \\ 0 & 6l_e^2 & 0 & -l_e^2 \\ -36 & 0 & 36 & -6l_e \\ 6l_e & -l_e^2 & -6l_e & 2l_e^2 \end{bmatrix} \tag{6.19}$$

$$[A_3] = \frac{1}{140l_e}\begin{bmatrix} 24 & -2l_e & -24 & 5l_e \\ -2l_e & 6l_e^2 & 2l_e & -l_e^2 \\ -24 & 2l_e & 24 & -5l_e \\ 5l_e & -l_e^2 & -5l_e & \frac{4}{3}l_e^2 \end{bmatrix} \tag{6.20}$$

$$[B_1] = \frac{EAC_o}{3EAl_e}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} \tag{6.21}$$

$$[B_2] = \frac{EAC_o}{6EAl_e}\begin{bmatrix} 11 & -12 & 1 \\ -12 & 16 & -4 \\ 1 & -4 & 3 \end{bmatrix} \tag{6.22}$$

$$[B_1] = \frac{EAC_o}{3EAl_e}\begin{bmatrix} 23 & -26 & 3 \\ -26 & 32 & -6 \\ 3 & -6 & 3 \end{bmatrix} \tag{6.23}$$

Chapter 7

RESULTS AND DISCUSSION

The first step in any aeroelastic response and stability analysis is the evaluation of natural frequencies of the rotor blade. Using the inertial and structural model developed in this study, a structural dynamic analysis was performed. It may be noted that the inertial and the structural operators given in Eqs. 6.6 and 6.8 respectively, are nonlinear. Since the structural dynamic analysis requires only linear terms, all the nonlinear terms are neglected. The corresponding linear equation for one beam finite element can be written as:

$$[M]_i\{\ddot{q}\}_i + [K]_i\{q\}_i = 0 \tag{7.1}$$

where $[M]_i$ represents the mass matrix of i^{th} element, (given in Eq. 6.6. and in Sec. 6.2.1). The stiffness matrix $[K]_i$ consists of three components. They are $[K^{cf}]$ (given in Sec. 6.2.2), $[K^E]$ (given in Sec. 6.3.1) and $[K^{E'}]$ (given in Eq. 6.17). The element stiffness matrix is given as:

$$[K]_i = [K^{CF}] + [K^F] + [K^{E'}] \tag{7.2}$$

The element matrices are assembled to form the global finite element model for the rotor blade. For the tip element, the corresponding local to global transformation (Ref. 30) is performed to take into account the sweep and anhedral angles. The

transformation matrix $[\Lambda^c]$ is given in Appendix A. Imposing the root boundary conditions, the corresponding rows and columns from the global model are eliminated. The resulting matrix equation can written as:

$$[M]\{\ddot{q}\} + [K]\{q\} = 0 \tag{7.3}$$

Performing an eigen analysis, the natural frequencies of an undamped rotating blade in vacuum can be evaluated.

7.1 VALIDATION

In order to validate the finite element blade model developed in this study, the results of the present analysis are compared (for certain specific cases) with those available in the literature. The data shown in Table 1, correspond to a uniform, untwisted straight blade for which uncoupled natural frequencies are available in the literature (Refs. 24,26 and 27). The terminology soft-in-plane blade configuration indicates that the first non-dimensional rotating lag frequency is less than 1.

Using the data given Table 1, the uncoupled natural frequencies of the rotating blade are calculated. A convergence study on the number of elements is also performed. The results of the convergence study along with the results of Refs. 24,26 and 27 are shown in Table 2. The results of the present analysis are in excellent agreement with those available in literature. The natural frequencies converge to a good accuracy with 10 elements.

In all the subsequent calculations only 10 elements are considered. The effects varying presweep (β_s), precone (β_p) and predroop (β_d) on the natural frequencies of the straight rotor blade are shown in Table 3. The natural frequencies in flap, lag and torsional modes reduce with pre sweep angle, whereas they are not influenced by precone and predroop angles.

7.2 EFFECTS OF TIP SWEEP AND ANHEDRAL

The influence of tip sweep and anhedral angles on the natural frequencies and mode shapes of the rotor blade are analysed systematically. The blade data is given in Table 1. The blade is divided into 10 finite elements. The swept tip portion of the blade is taken as one element having a length 0.1l. The sweep and anhedral angles are varied from -30^o to $+30^o$ in steps of 5 degrees. For each case the coupled natural frequencies and mode shapes are evaluated. In total, there are 169 cases. The frequencies corresponding to all these cases are given in Appendix B, for reference and for future validation purposes.

The variation of natural frequencies with sweep angles are shown in 3-D plots in Figs. 12 to 17. Figure 12 shows the variation of first lag frequency with sweep angles. The results show that depending on the value of anhedral angle, the first lag frequency decreases monotonically with sweep angle (for $\Lambda_a = +30^o$) or decreases and increases with sweep angles (for $\Lambda_a = -30^o$). The first lag frequency varies from 0.8898 ($\Lambda_a = 30^o, \Lambda_s = -30^o$) to 0.6892 ($\Lambda_a = 25^o, \Lambda_s = 30^o$). A similar nature is also observed for second lag frequency, as shown in Fig. 13.

The variation of first flap frequency with sweep angles is shown in Fig. 14. It is interesting to note that depending on the value of anhedral angle, the flap frequency increases monotonically (for $\Lambda_a = +30^o$) or decreases monotonically (for $\Lambda_a = -30^o$) with sweep angles. The influence of sweep angle is more pronounced for large values of anhedral angles. The first flap frequency varies from 1.2036 ($\Lambda_a = \Lambda_s = 30^o$) to 1.1003 ($\Lambda_a = 30^o, \Lambda_s = -30^o$). Similar behaviour is also seen for second flap frequency, shown in Fig. 15. The third flap frequency exhibits a different phenomenon ith variation in sweep and anhedral angles (Fig. 16). The reason may be due to complex modal coupling with lag and torsional modes. The torsional frequency

seems to be less influenced by sweep and anhedral angles as shown in Fig. 17.
It is found from the results that anhedral angle alone introduces lag-torsion coupling and sweep angle introduces flap-torsion coupling. The influence tip anhedral angle on the mode shape in first lag is shown in Fig. 18. The torsional mode contribution is increasing with increase in the value of Λ_a. The flap motion has negligible contribution in first lag mode.
The influence of tip sweep on first flap mode is shown in Fig. 19. Even in this case, the participation of torsional motion increases with increase in Λ_s. The participation of lag motion is negligible.
The combined effect of both anhedral and sweep angles on the mode shapes are shown in Figs. 20-25. The combination of sweep and anhedral angles introduce significant lag-flap-torsion coupling in first lag mode (Fig. 20), whereas the flap contribution is negligible in second lag mode (Fig. 21). The contribution of lag mode is negligible in all the flap modes (Figs. 22-24) even with influence of Λ_a. Torsional mode shape is unaffected by sweep and anhedral angles as shown in Fig. 25.
To quantify the lag-torsion and flap-torsion coupling in various modes, the value of torsional deformation at 0.75 l ($\Phi_{0.75}$) is taken as the reference parameter. (Since this cross-section is treated as a typical section for aerodynamic load calculations in a rotor blade). The variation $\Phi_{0.75}$ as a function of sweep angles is shown in Figs. 26-30 for the lag and flap modes.
In lag modes, lag-torsion coupling is significantly influenced by anhedral angle compared to sweep angle as shown in Figs. 26 and 27. In first flap mode, the sweep angle has a significant influence on flap-torsion coupling compared to anhedral angle. In second flap mode, both anhedral and sweep angles influence the flap-torsion coupling. A similar variation is observed in third flap mode but of a much smaller magnitude.

Chapter 8

CONCLUDING REMARKS

In this thesis, finite element model of a rotor blade with swept tip has been developed for the dynamic analysis of the blade. The model includes all the geometric complexities of the hub and hub motion.

The equations of motion have been derived using Hamilton's principle . The general equations are then specialised for the structural dynamic analysis of the rotor blade. The formulation was validated by comparing the results of the present analysis (for a straight blade) with those available in literature.

The influence of tip sweep on natural frequencies and mode shapes has been analysed systematically. Anhedral angle introduces significant lag-torsion coupling and sweep angle introduces flap-torsion coupling.

Table 1: Input data for soft-in-plane blade

Soft-in-plane blade data
$Im_{\zeta\zeta}/ml^2 = 0.004$
$Im_{\eta\eta}/ml^2 = 0.000$
$\theta_G = 0$
$m = 1$
$\beta_s = 0$
$\beta_d = 0$
$\beta_p = 0$
$\theta_I = 0$
$GJ/m\Omega^2l^4 = 0.001473$
$EI/m\Omega^2l^2 = 20.0$
$e_1 = 0.0$
$e_2 = 0.0$
$a = 0.0$
$EAC_o/EA = 0.00021036$
$EI_{\zeta\zeta}/m\Omega^2l^4 = 0.0301$
$EI_{\eta\eta}/m\Omega^2l^4 = 0.0106$

Offsets of mass centre and aerodynamic centre from elastic centre are zero.

Table 2: Natural frequencies of soft-in-plane blade

Mode	Uncoupled Natural Frequencies						
	No. of elements				Ref. [26]	Ref. [27]	Ref. [24]
	N=2	N=5	N=10	N=20			
1st Lag	0.73519	0.73128	0.73114	0.73113	0.7326	0.7317	0.732
2nd lag	4.49322	4.45488	4.45317	4.45306	4.4563	4.4825	-
1st Flap	1.13260	1.12697	1.12578	1.12513	1.1247	1.1245	1.125
2nd Flap	3.45728	3.44530	3.44239	3.42668	3.4089	3.4073	-
3rd Flap	7.79626	7.77756	7.77462	7.71541	7.6376	7.6218	-
1st Torsion	3.26406	3.26330	3.26328	3.26328	3.2632	-	3.263
1st Axial	6.94918	6.94035	6.93925	6.93898	6.9533	-	-

Table 3: Influence of precone, predroop and preswept angles on natural frequencies

Mode	$\beta_s = 0$ $\beta_p = 0$ $\beta_d = 0$	$\beta_s = 2$ $\beta_p = 0$ $\beta_d = 0$	$\beta_s = 3$ $\beta_p = 0$ $\beta_d = 0$	$\beta_p = 2$ $\beta_s = 0$ $\beta_d = 0$	$\beta_d = 2$ $\beta_s = 0$ $\beta_p = 0$
1st Lag	0.7311	0.7034	0.6891	0.7311	0.7311
2nd Lag	4.4531	4.4280	4.4154	4.4531	4.4531
1st Flap	1.1257	1.1085	1.0998	1.1257	1.1257
2st Flap	3.4423	3.4088	3.3920	3.4423	3.4423
3st Flap	7.7746	7.7334	7.7127	7.7746	7.7746
1st Torsion	3.2632	3.2602	3.2587	3.2632	3.2632
1st Axial	6.9392	6.9392	6.9392	6.9392	6.9392

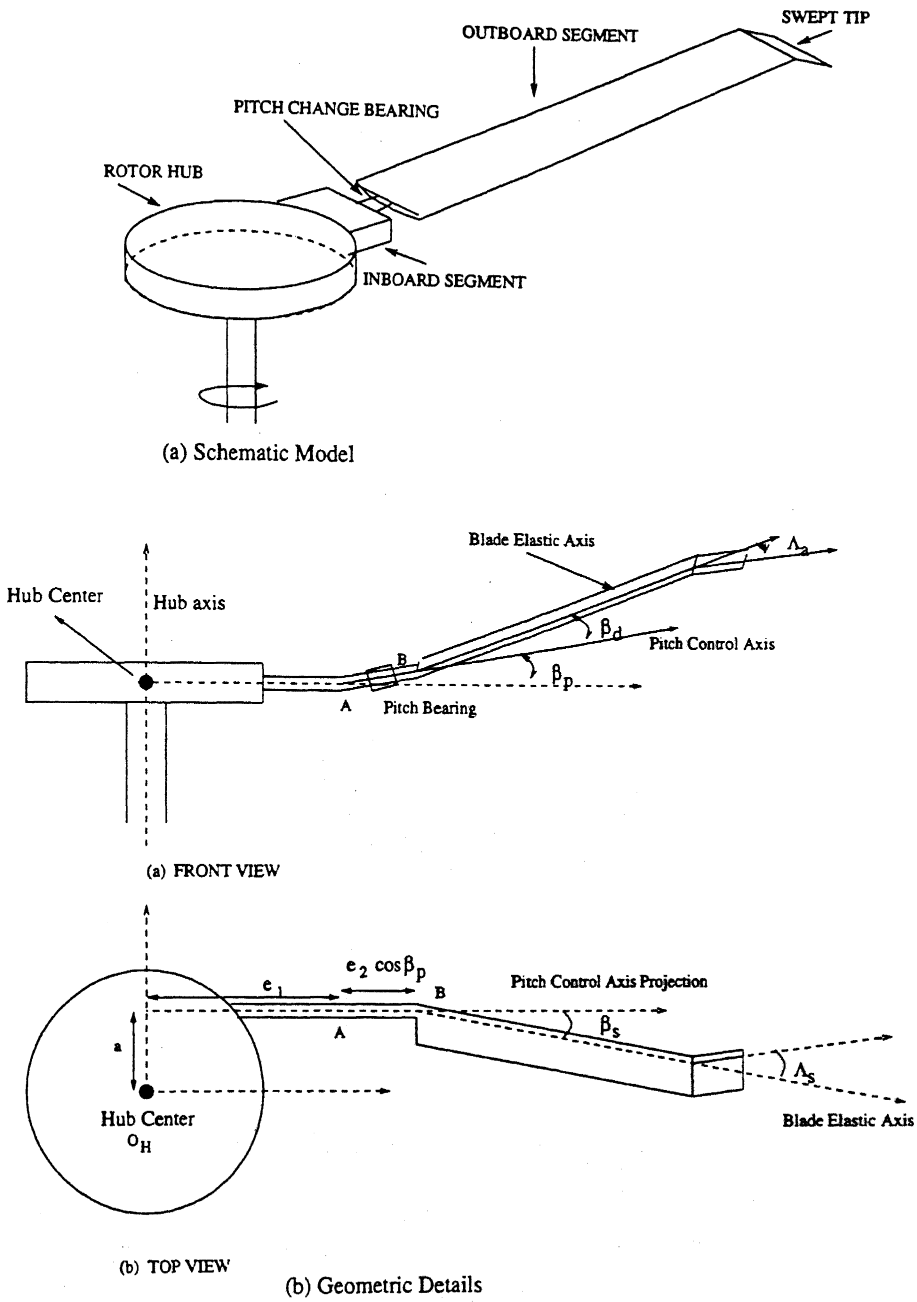


Figure 1: Rotor blade with tip sweep and anhedral

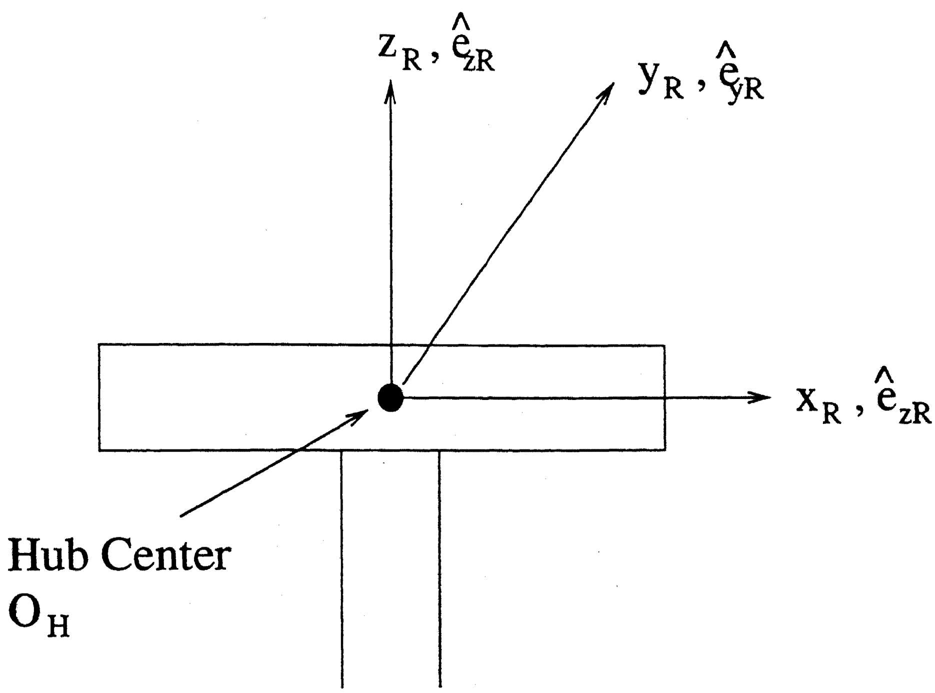


Figure 2: Inertial System-R

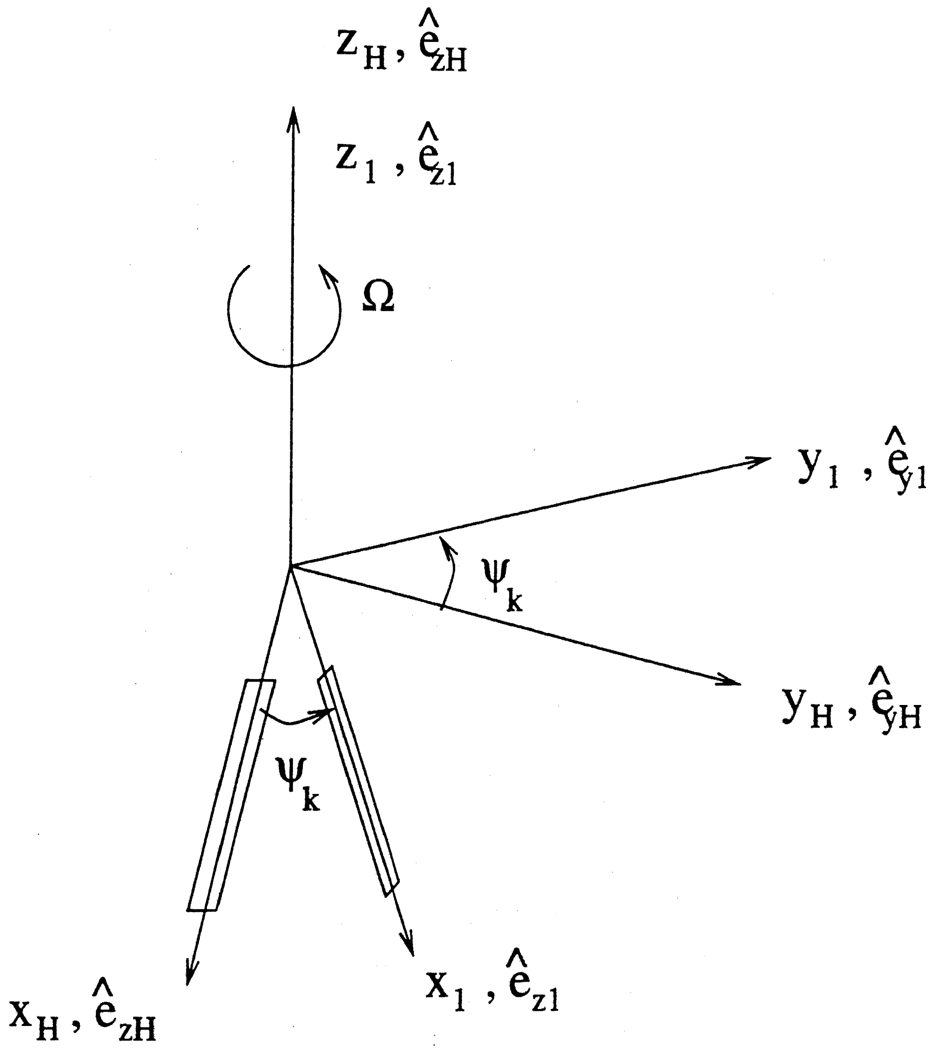


Figure 4: Rotating Hub System - 1

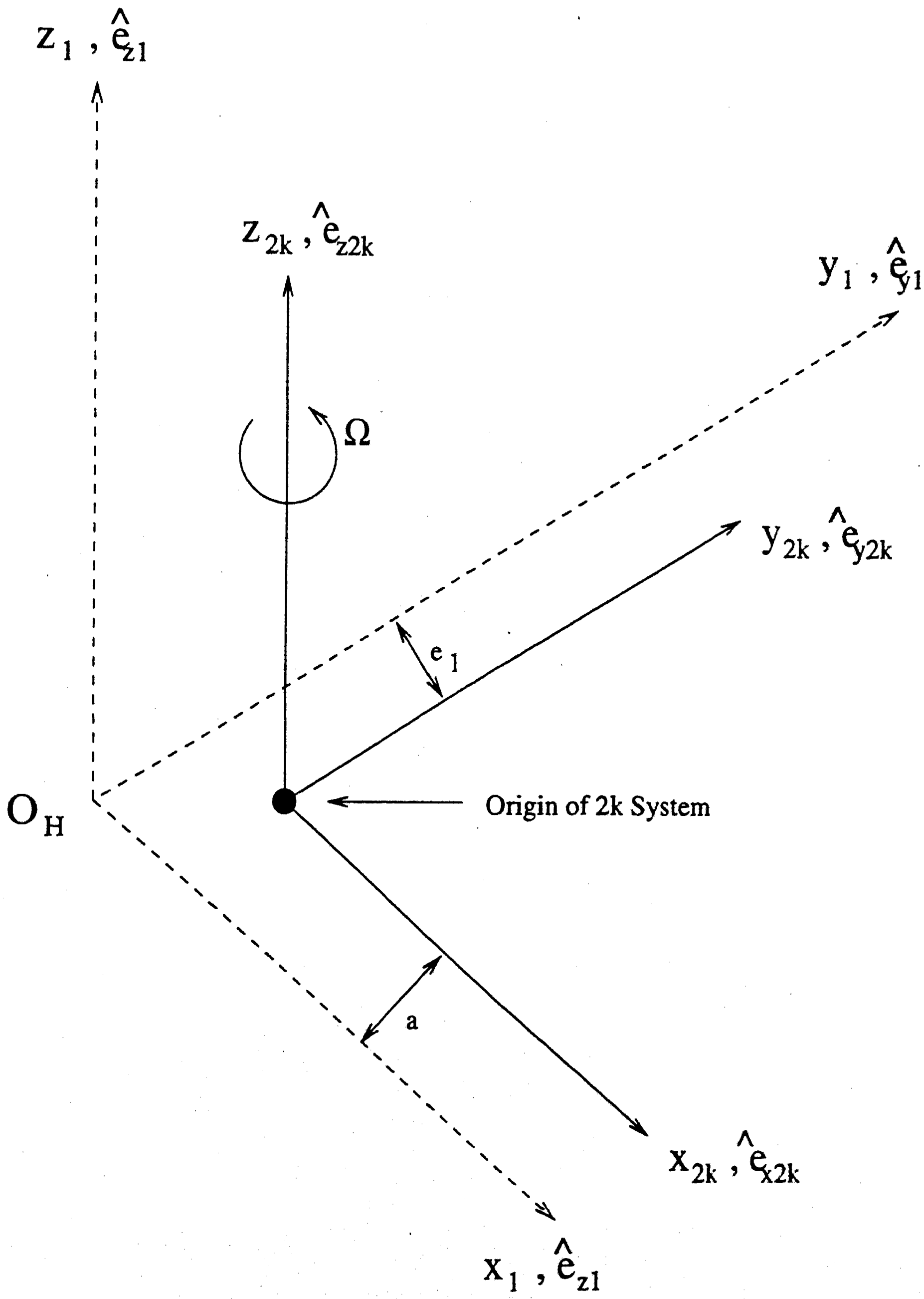


Figure 5: Rotating Hub System -2

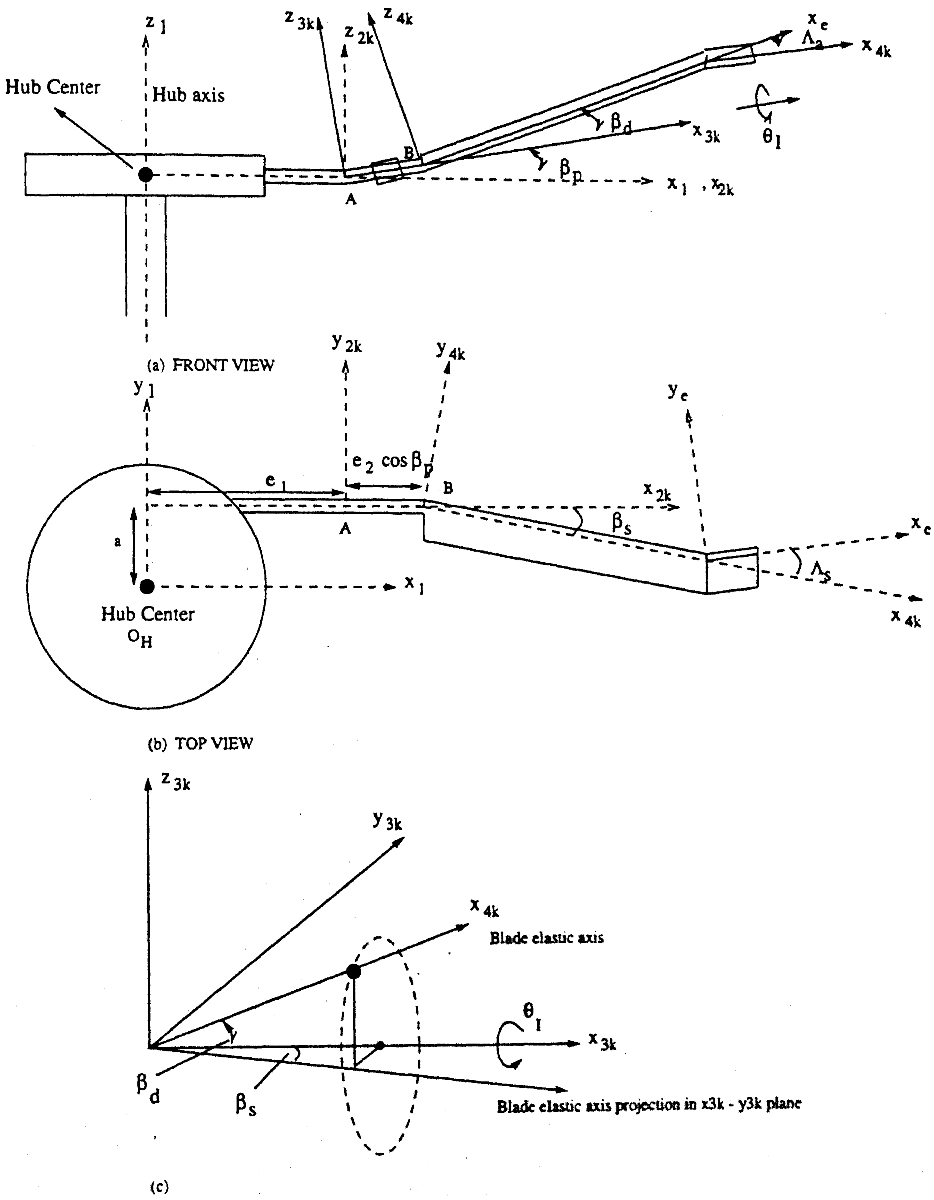


Figure 6: Blade Coordinate System 3k and 4k

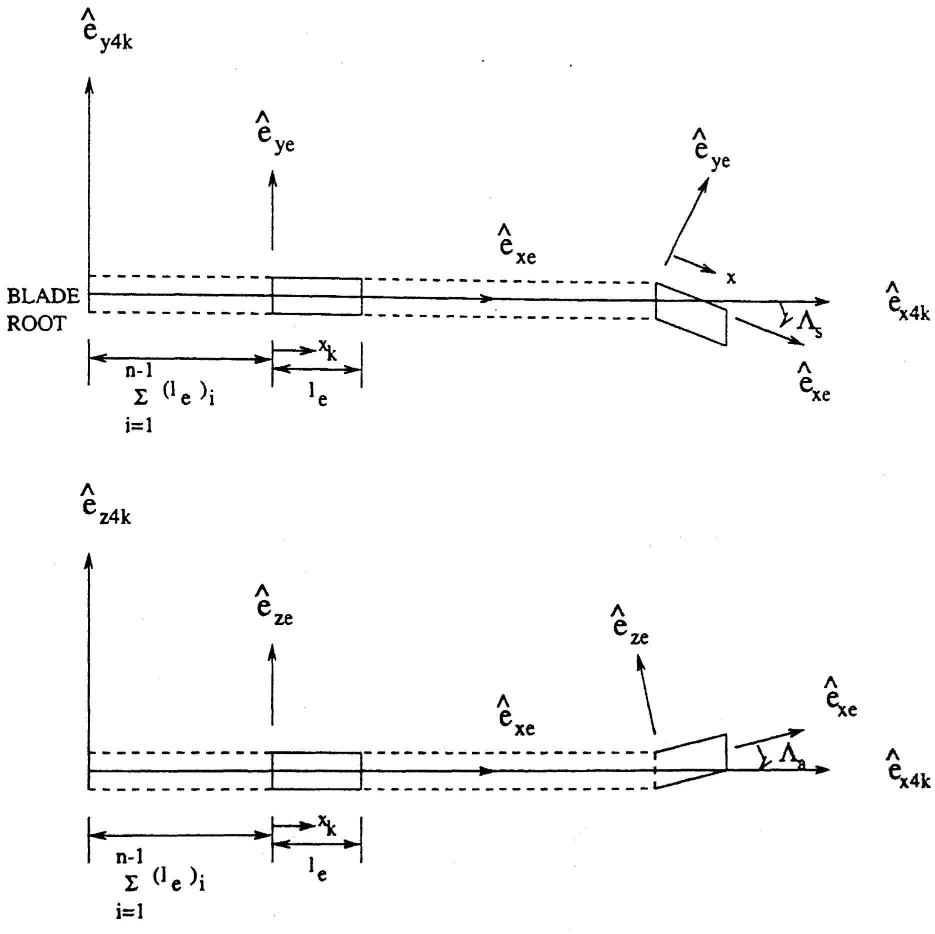


Figure 7: Undeformed Element Coordinate System

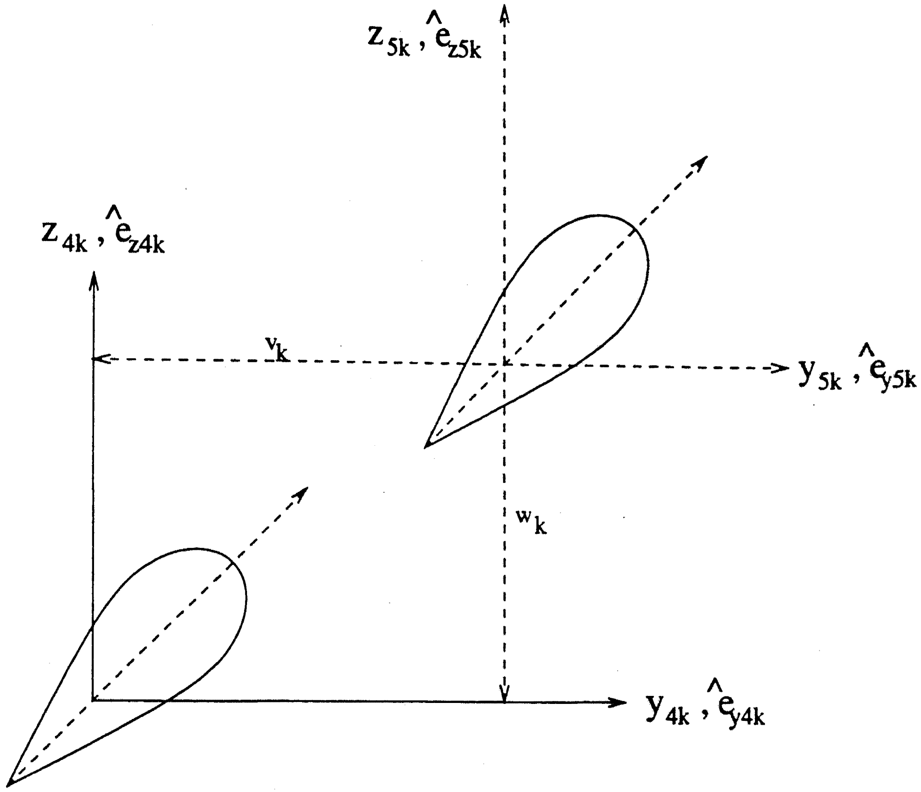


Figure 8: Rotating Blade Fixed System -5k

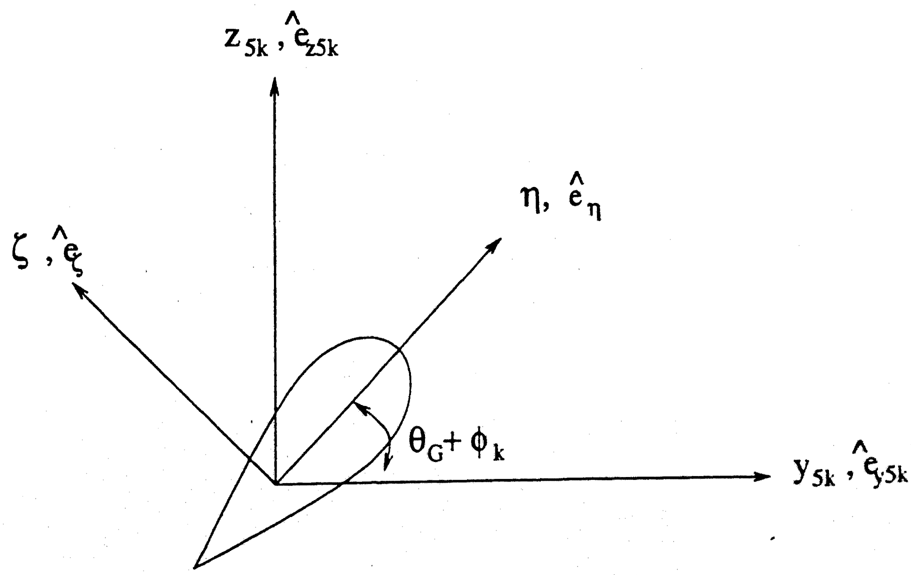


Figure 9: Cross-sectional Principal Coordinate System

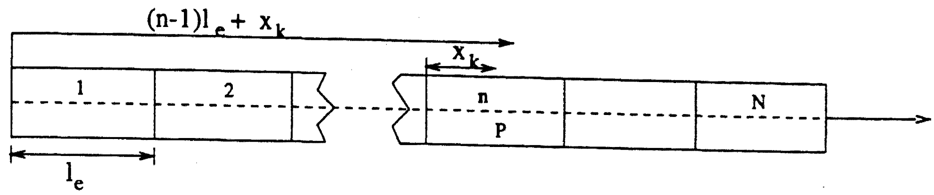


Figure 10: Finite Element Model of Blade

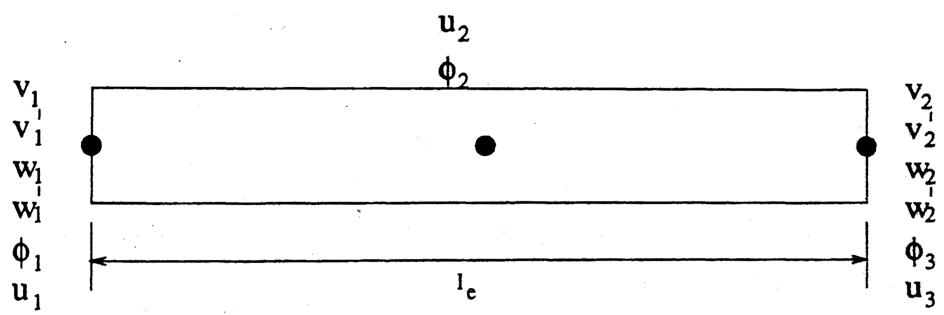


Figure 11: Element Nodal Degrees of Freedom

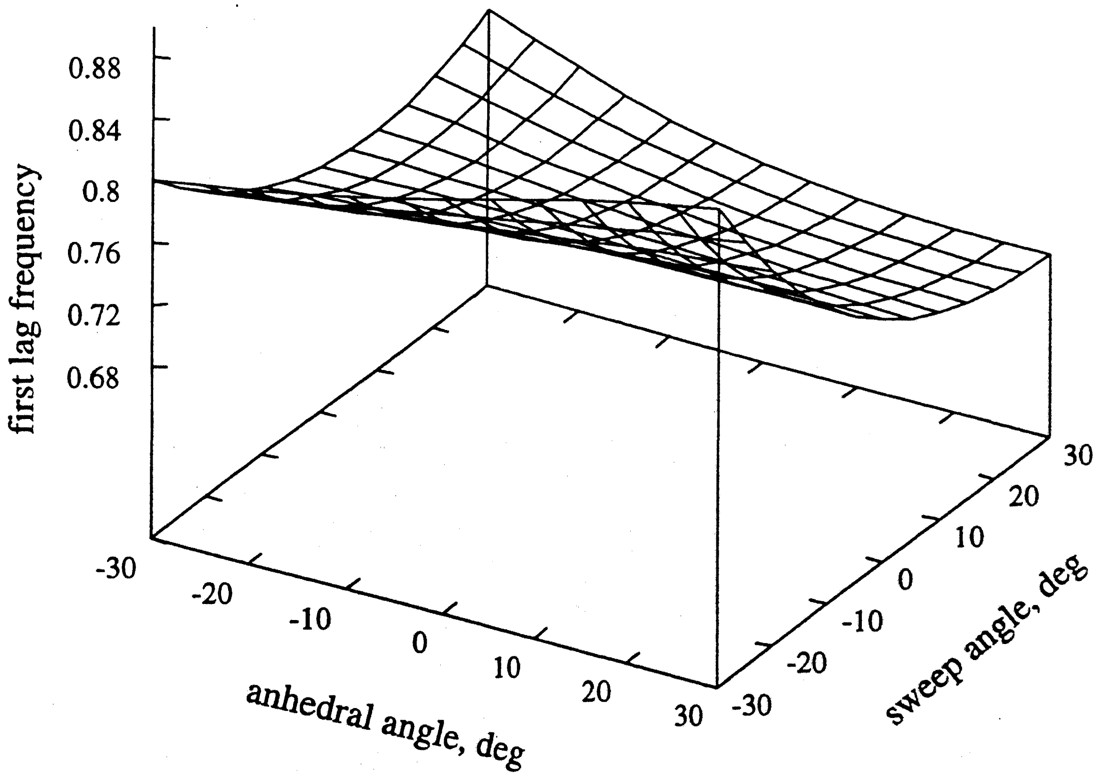


Figure 12: Non-dimensional first lag natural frequency as a function of tip sweep angle

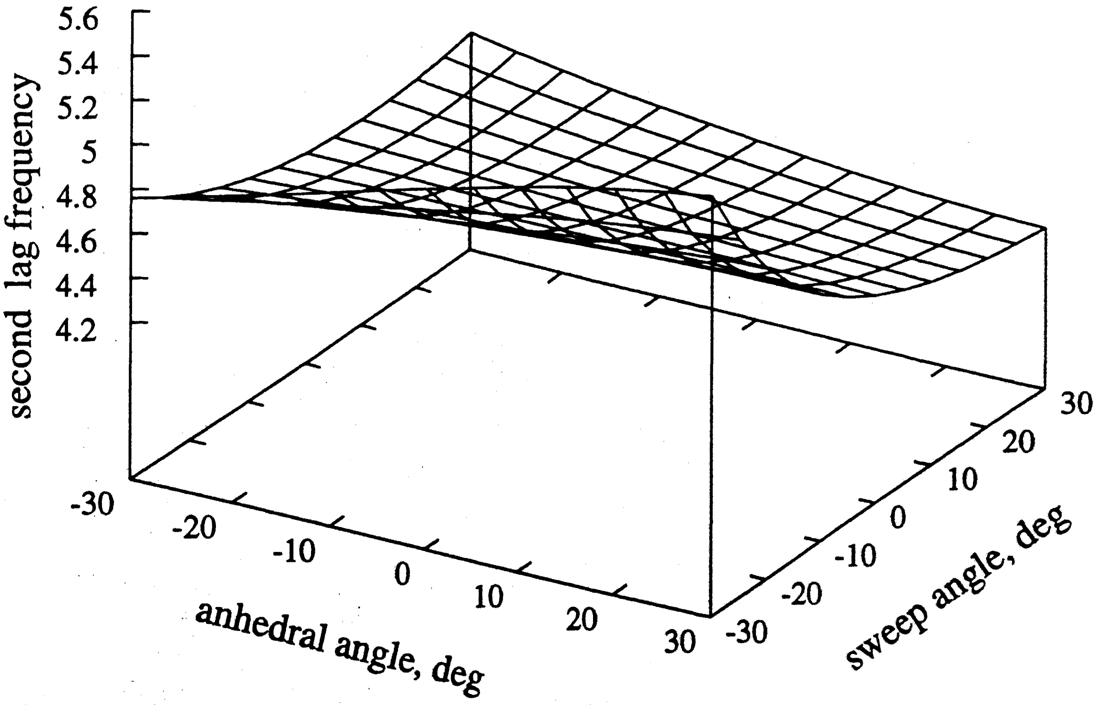


Figure 13: Non-dimensional second lag natural frequency as a function of tip sweep angle

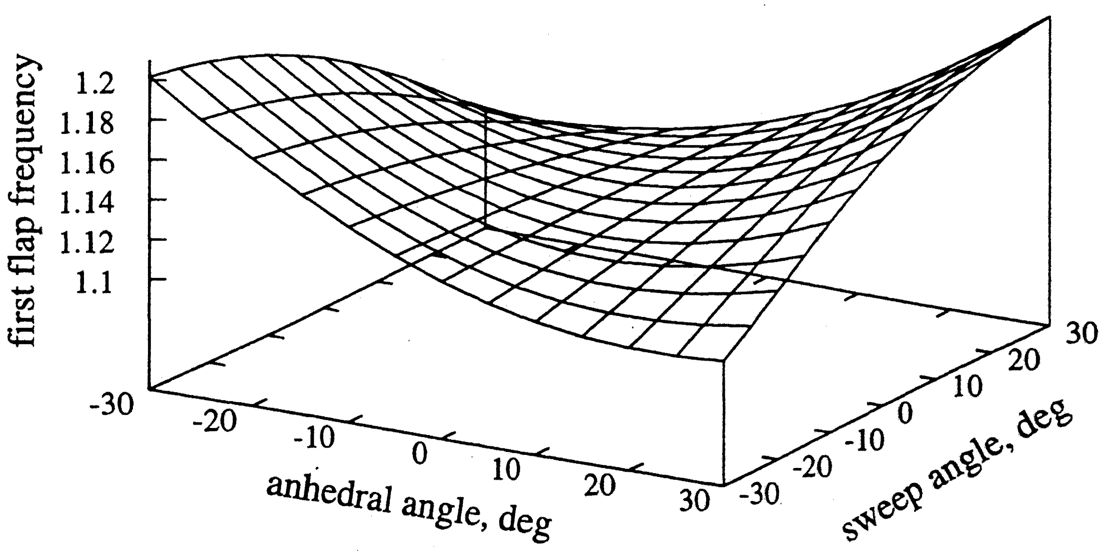


Figure 14: Non-dimensional first flap natural frequency as a function of tip sweep angle

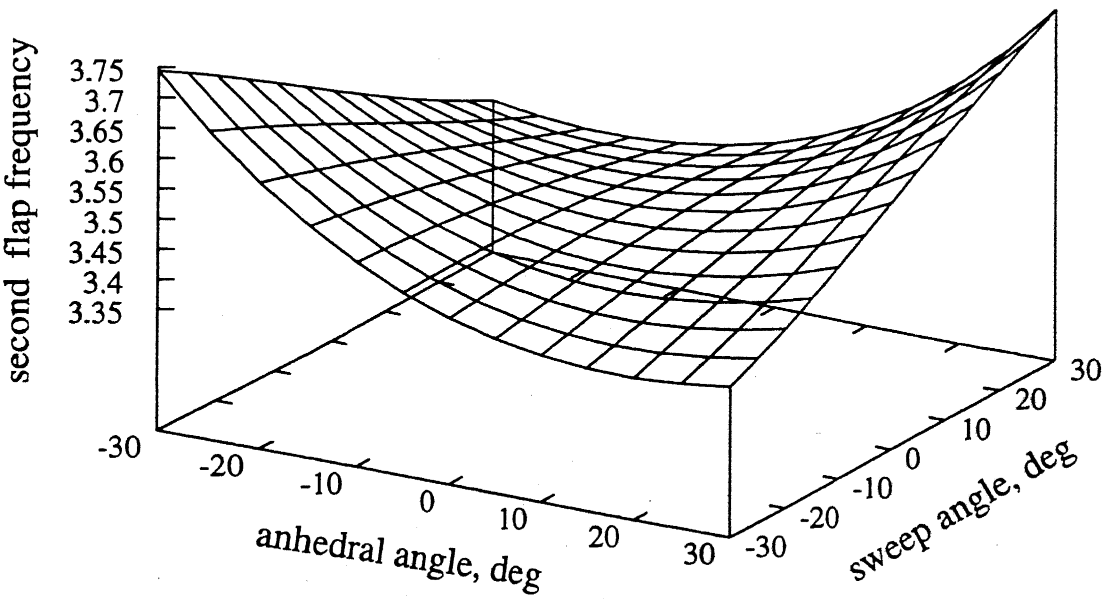


Figure 15: Non-dimensional second flap natural frequency as a function of tip sweep angle

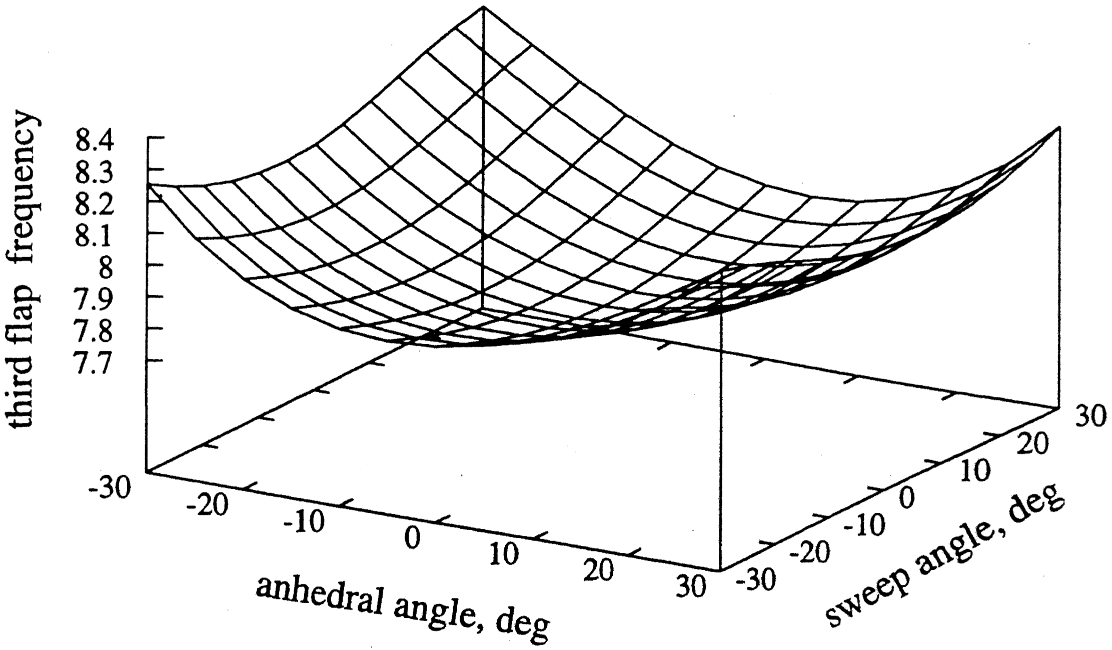


Figure 16: Non-dimensional third flap natural frequency as a function of tip sweep angle

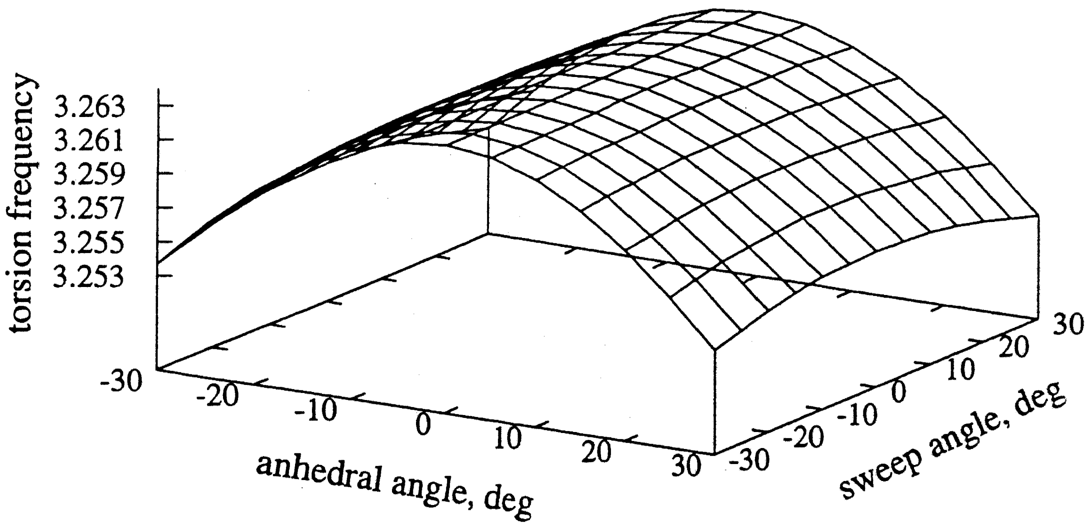
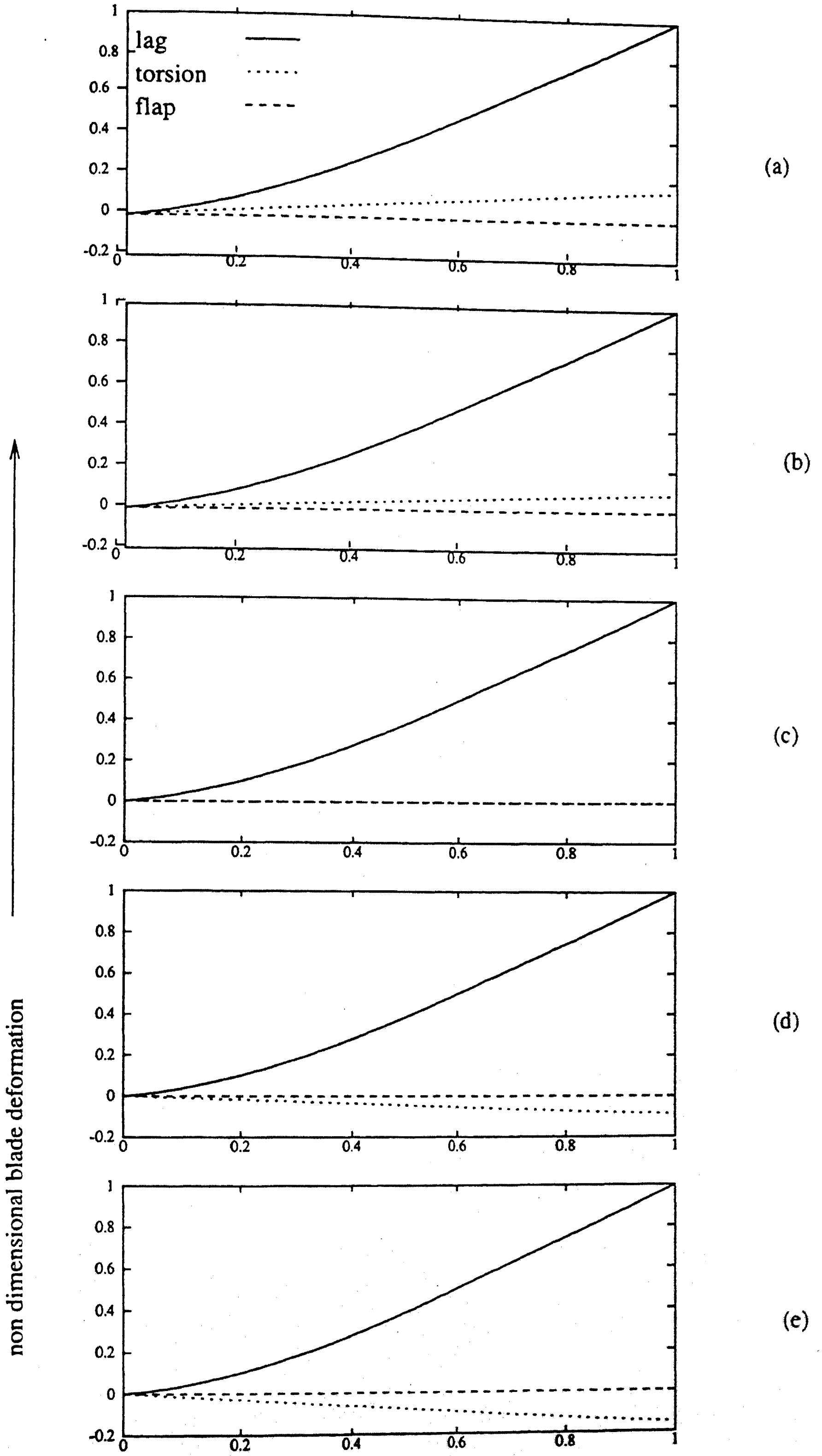


Figure 17: Non-dimensional torsion natural frequency as a function of tip sweep angle



Λ_a =- 30
$\Lambda_s = 0$

Λ_a =- 15
$\Lambda_s = 0$

$\Lambda_a = 0$
$\Lambda_s = 0$

Λ_a =15
$\Lambda_s = 0$

Λ_a =30
$\Lambda_s = 0$

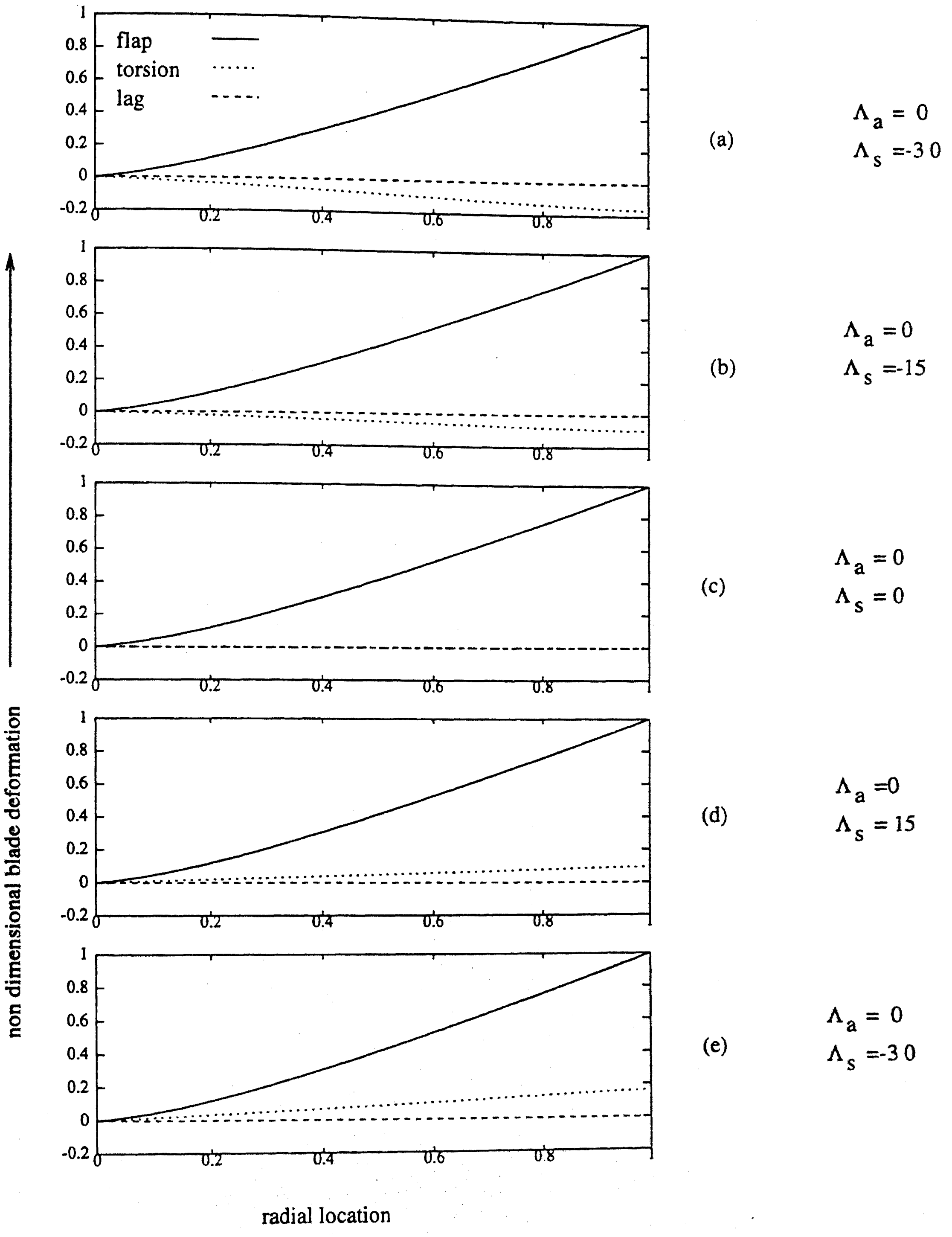


Figure 19: Coupled mode shape in first flap (Λ_a, Λ_s in degrees)

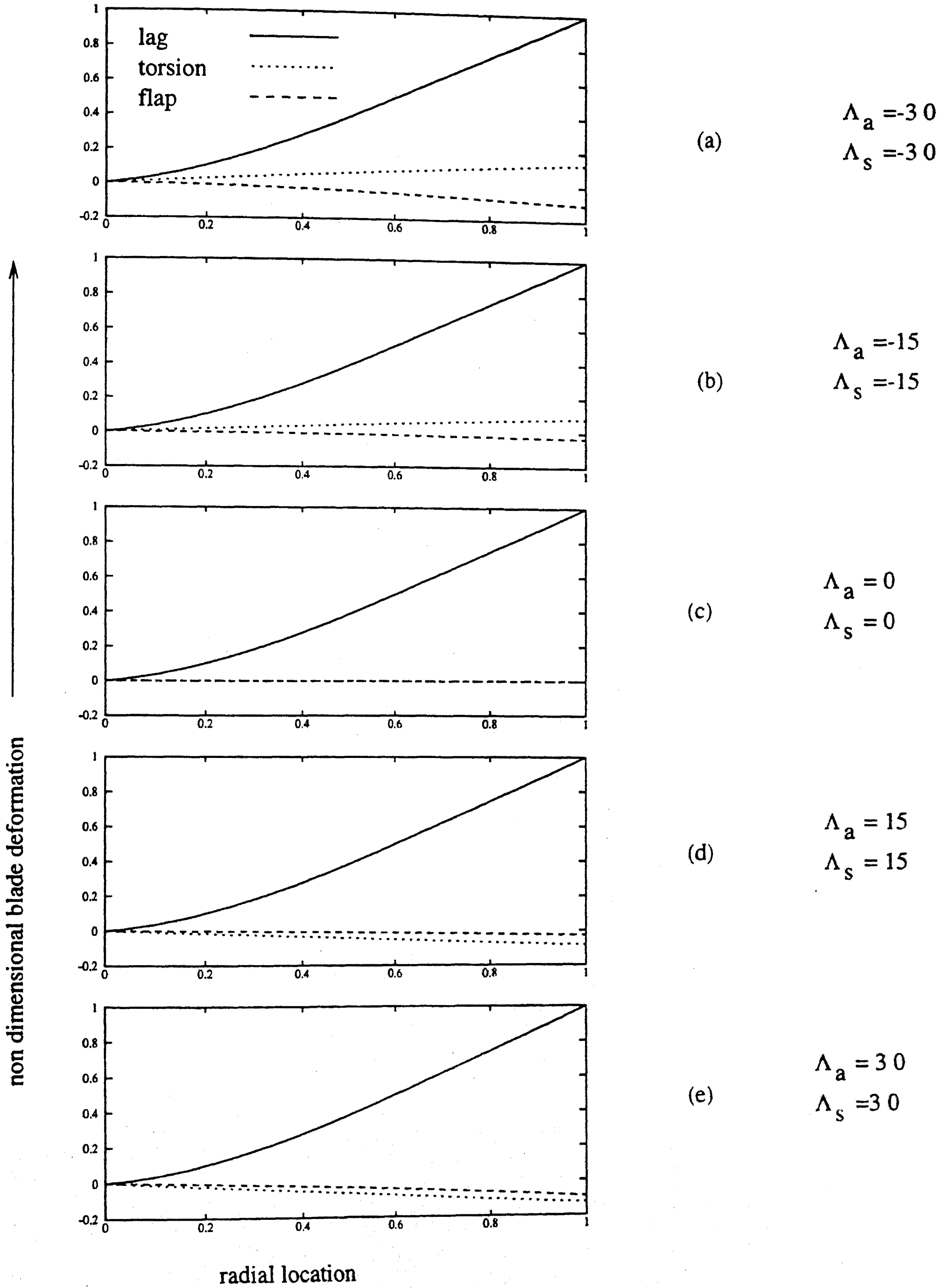


Figure 20: Coupled mode shape in first lag (Λ_a, Λ_s in degrees)

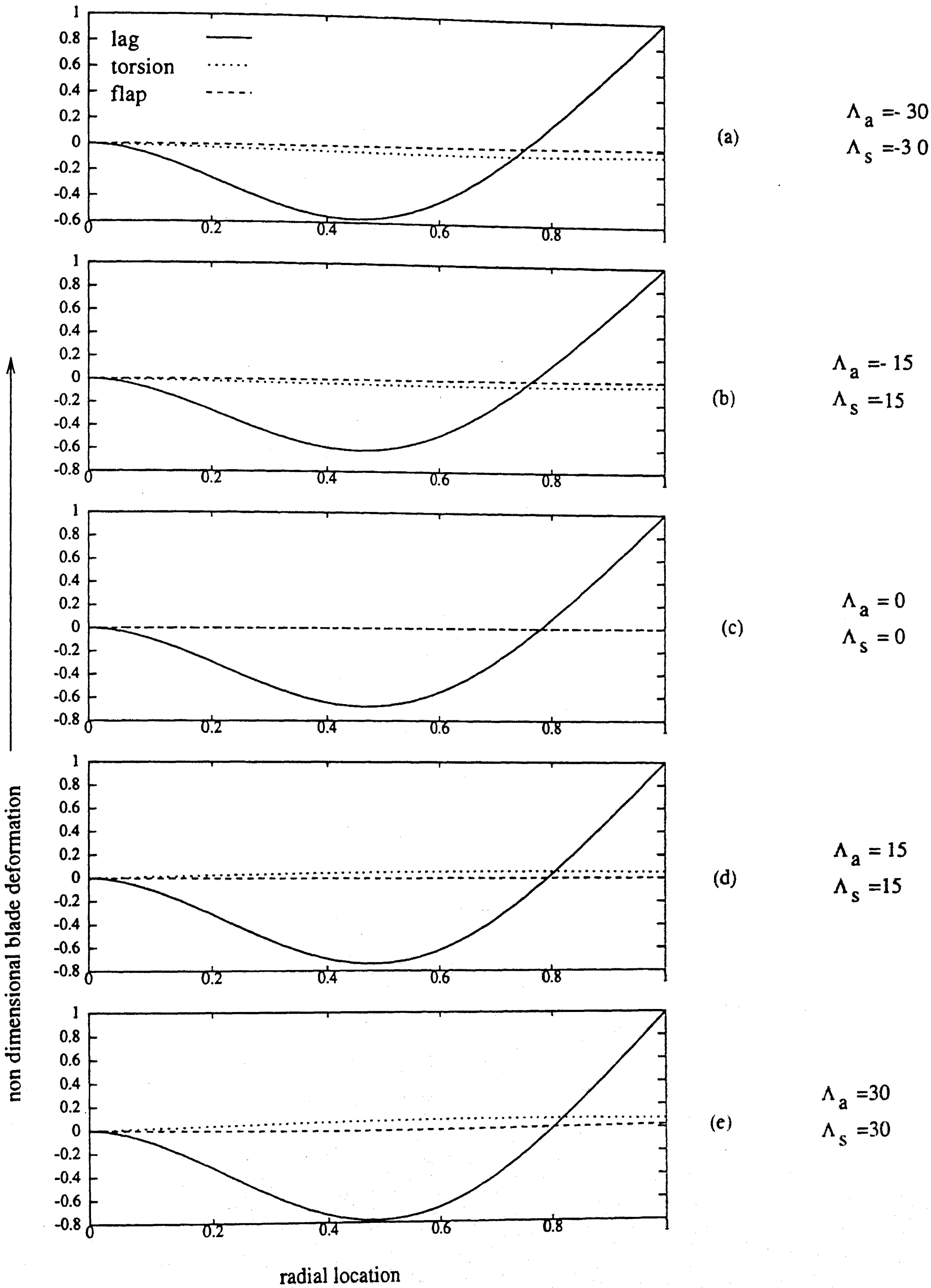
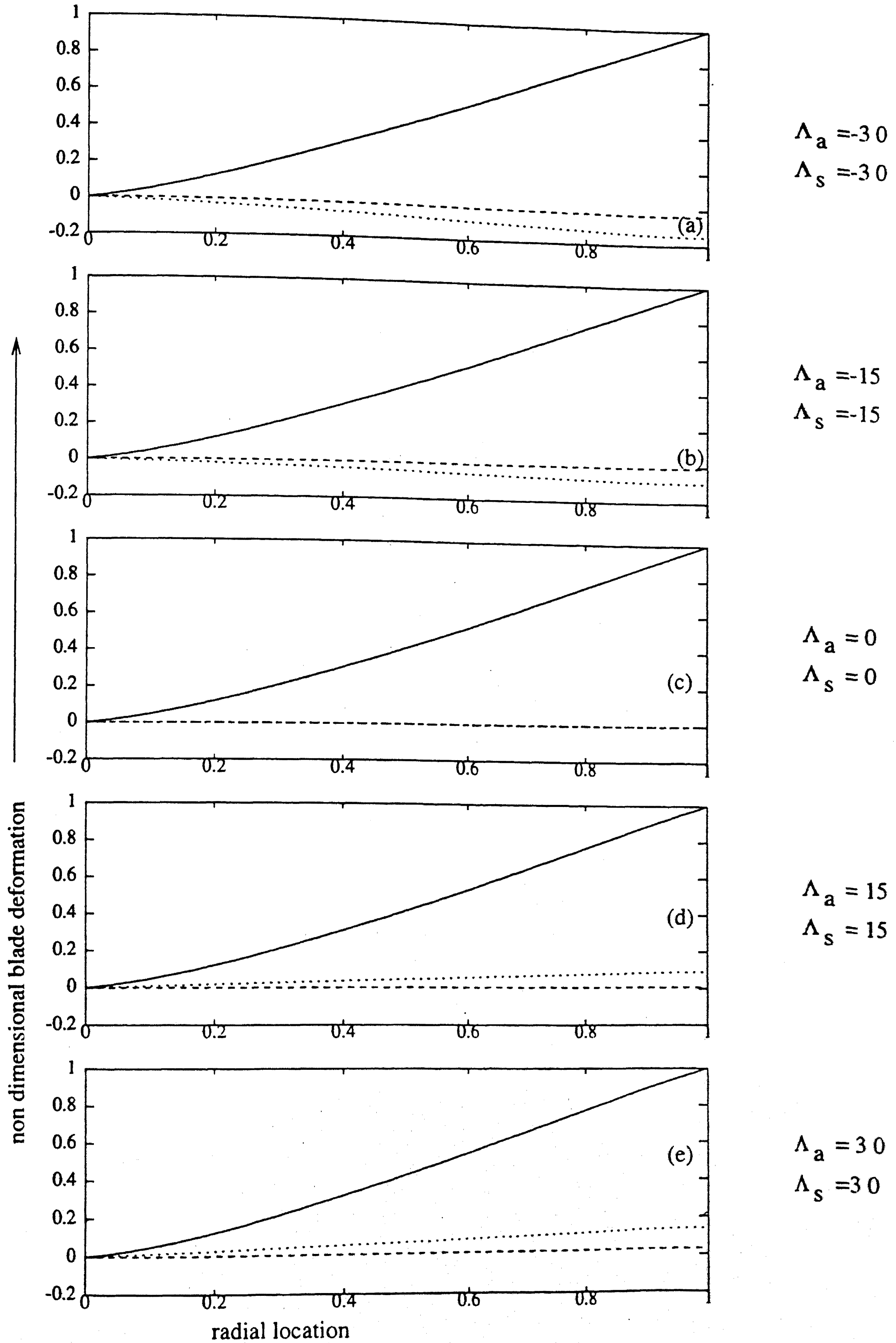


Figure 21: Coupled mode shape in second lag (Λ_a, Λ_s in degrees)



non dimensional blade deformation
radial location
1
0.8
0.6
0.4
0.2
0
-0.2
0
0.2
0.4
0.6
0.8
1
(a)
Λ_a =-3 0
Λ_s =-3 0
(b)
Λ_a =-15
Λ_s =-15
(c)
Λ_a = 0
Λ_s = 0
(d)
Λ_a = 15
Λ_s = 15
(e)
Λ_a = 3 0
Λ_s =3 0

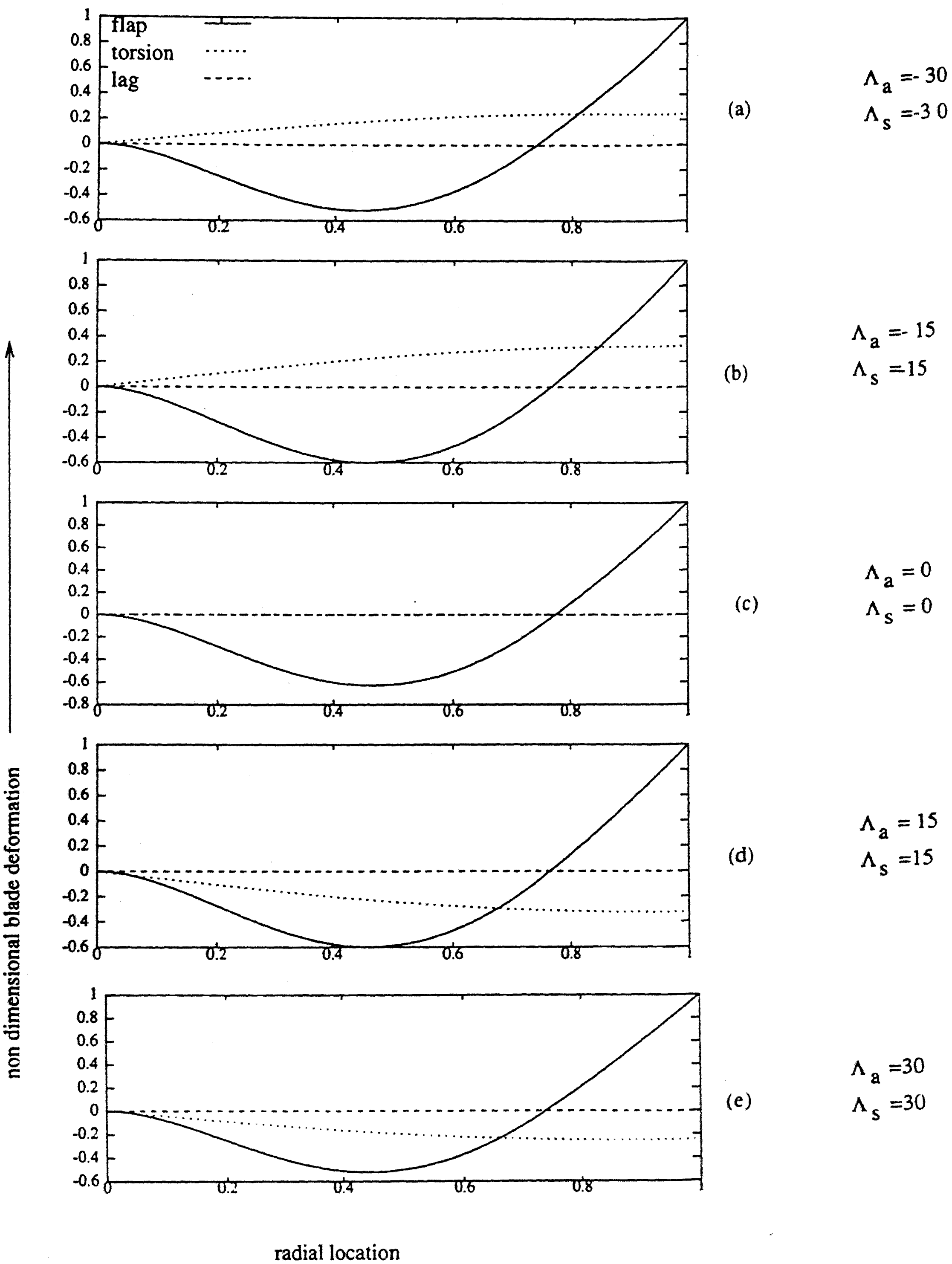


Figure 23: Coupled mode shape in second flap (Λ_a, Λ_s in degrees)

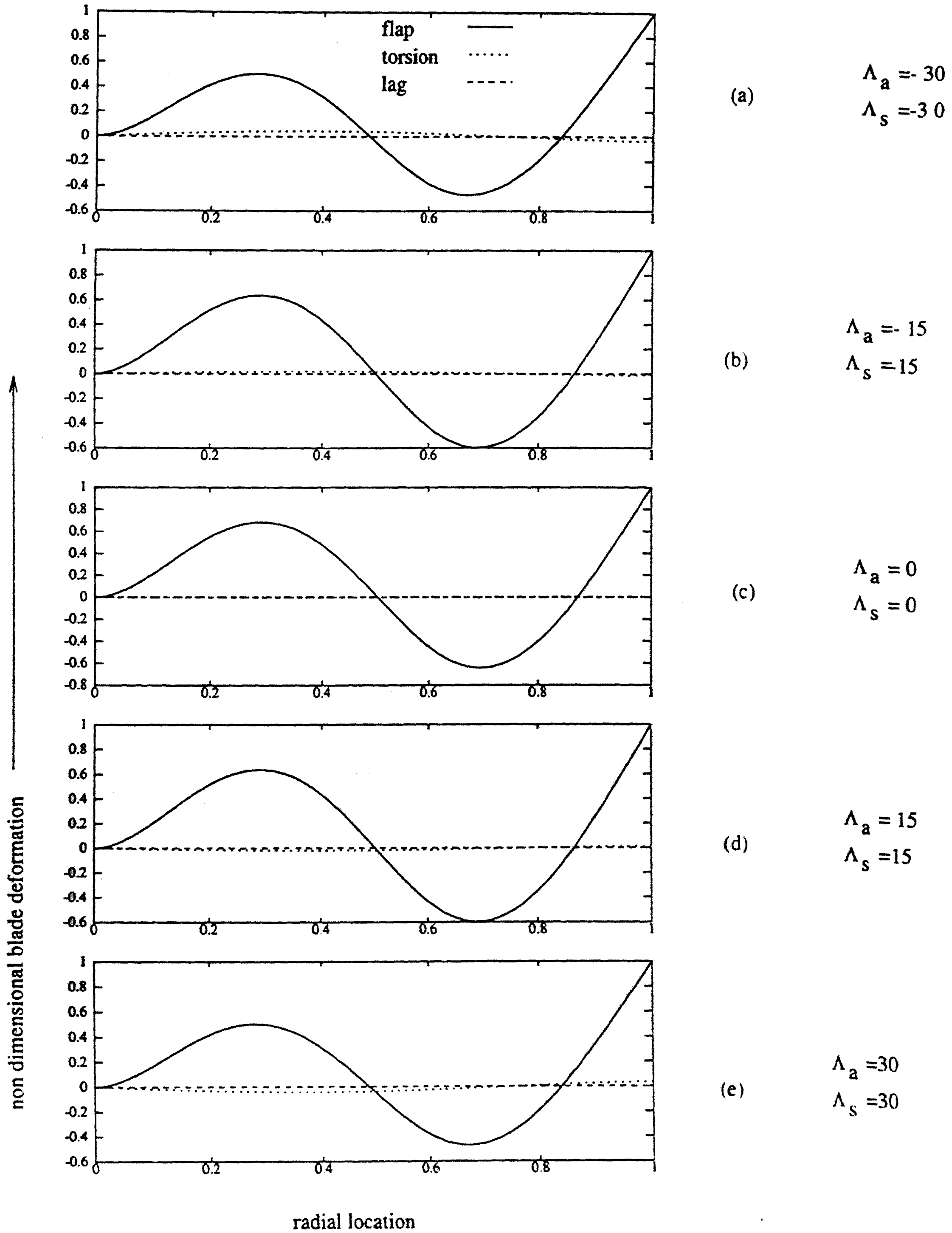


Figure 24: Coupled mode shape in third flap (Λ_a, Λ_s in degrees)

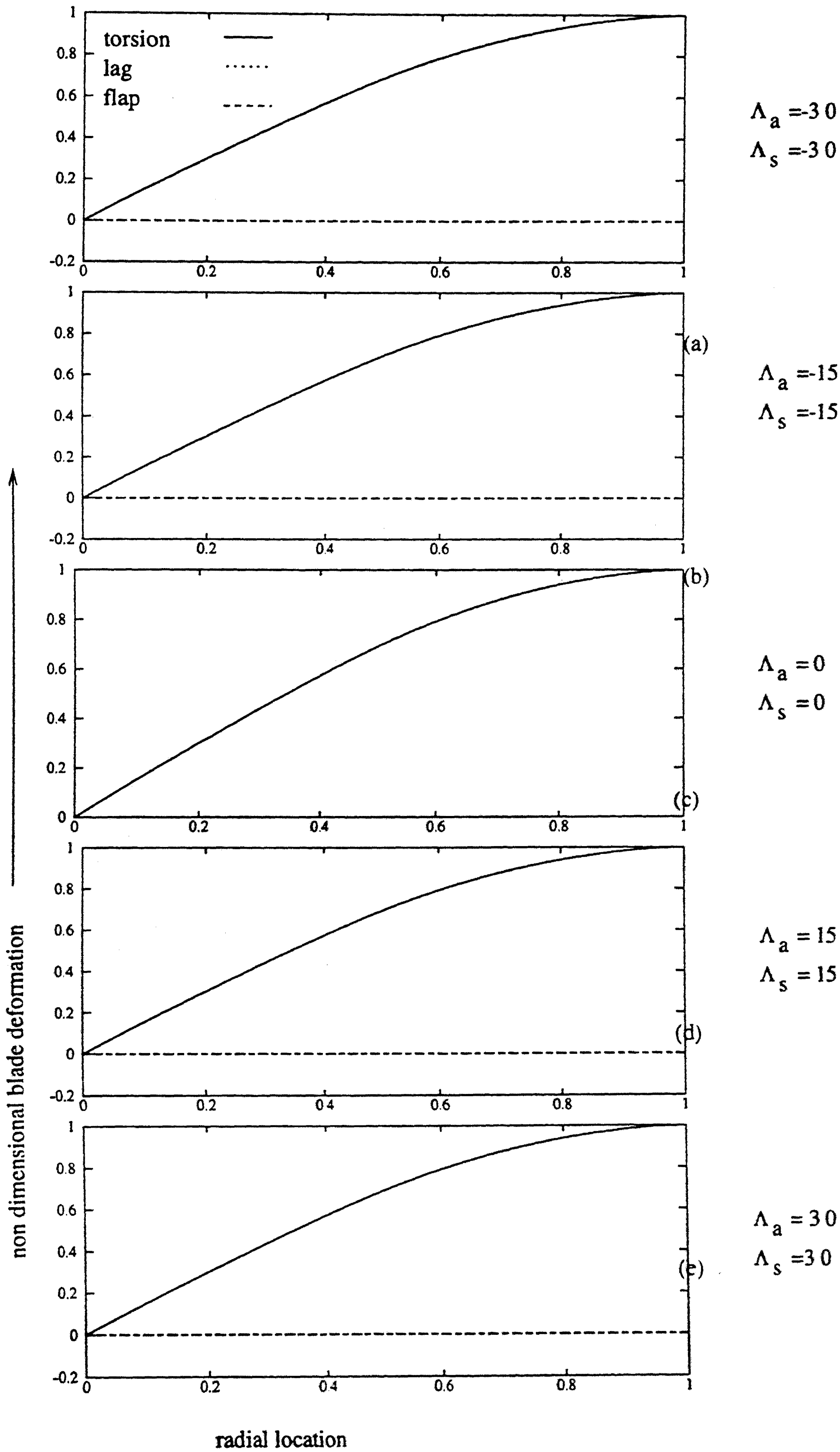


Figure 25: Coupled mode shape in torsion (Λ_a, Λ_s in degrees)

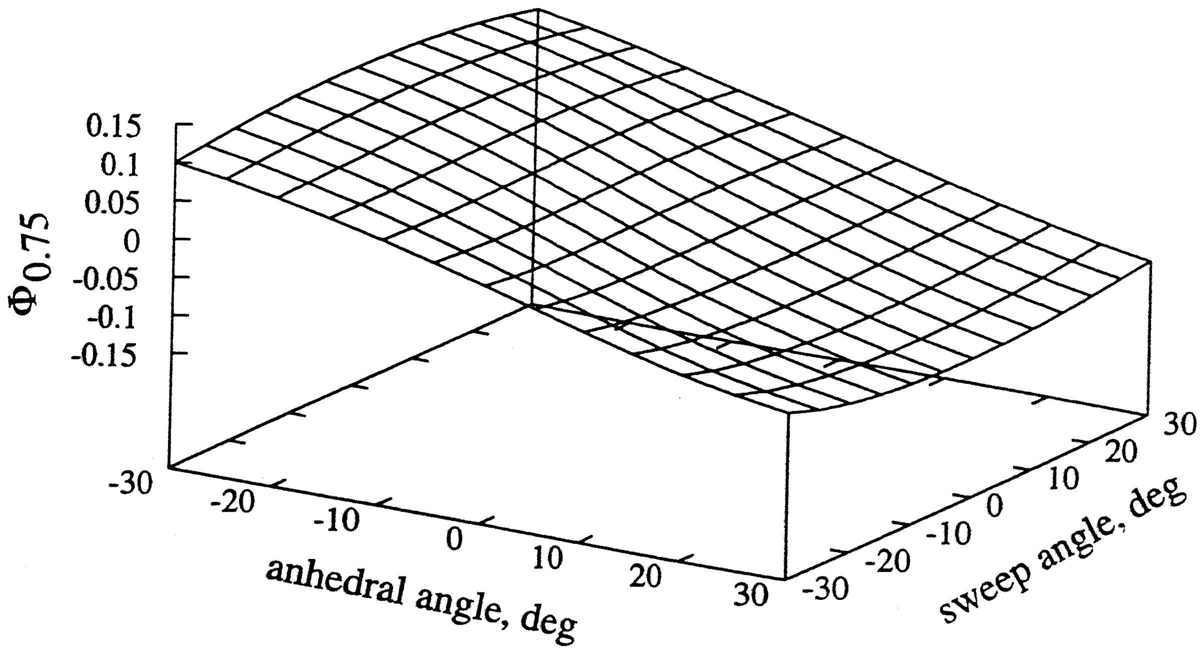


Figure 26: Lag-torsion coupling Vs tip sweep angle in first lag mode

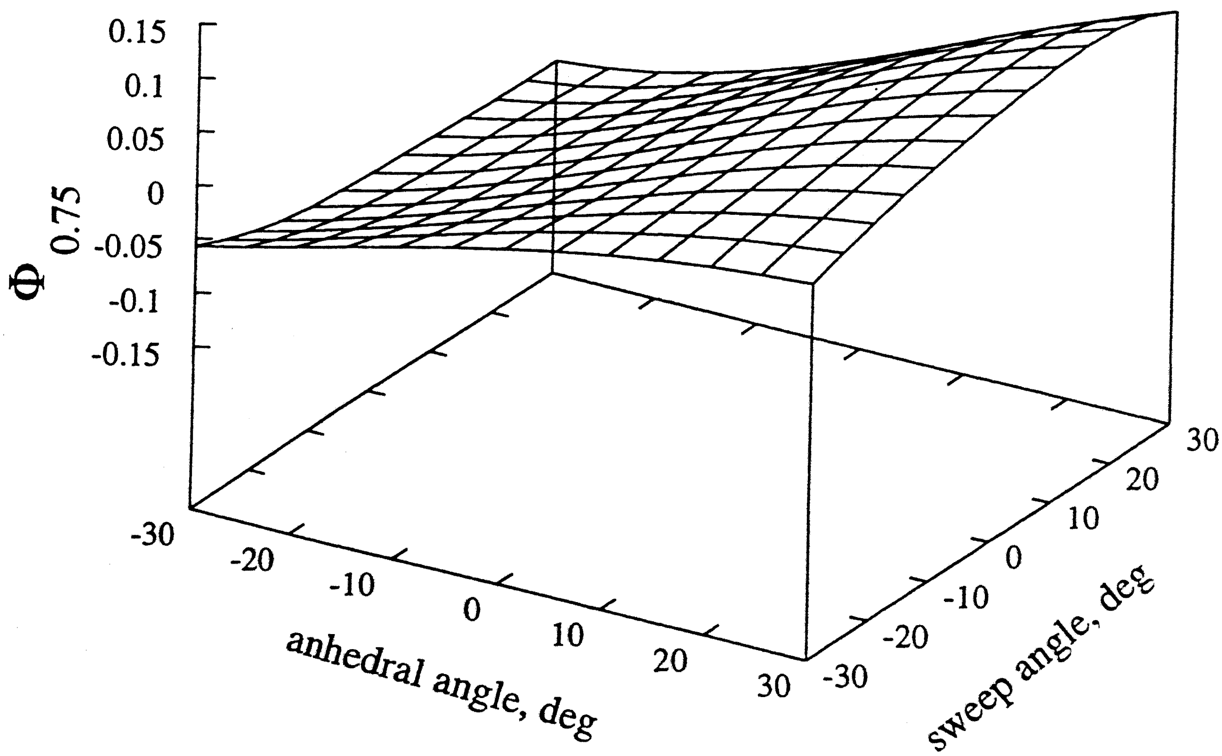


Figure 27: Lag-torsion coupling Vs tip sweep angle in second lag mode

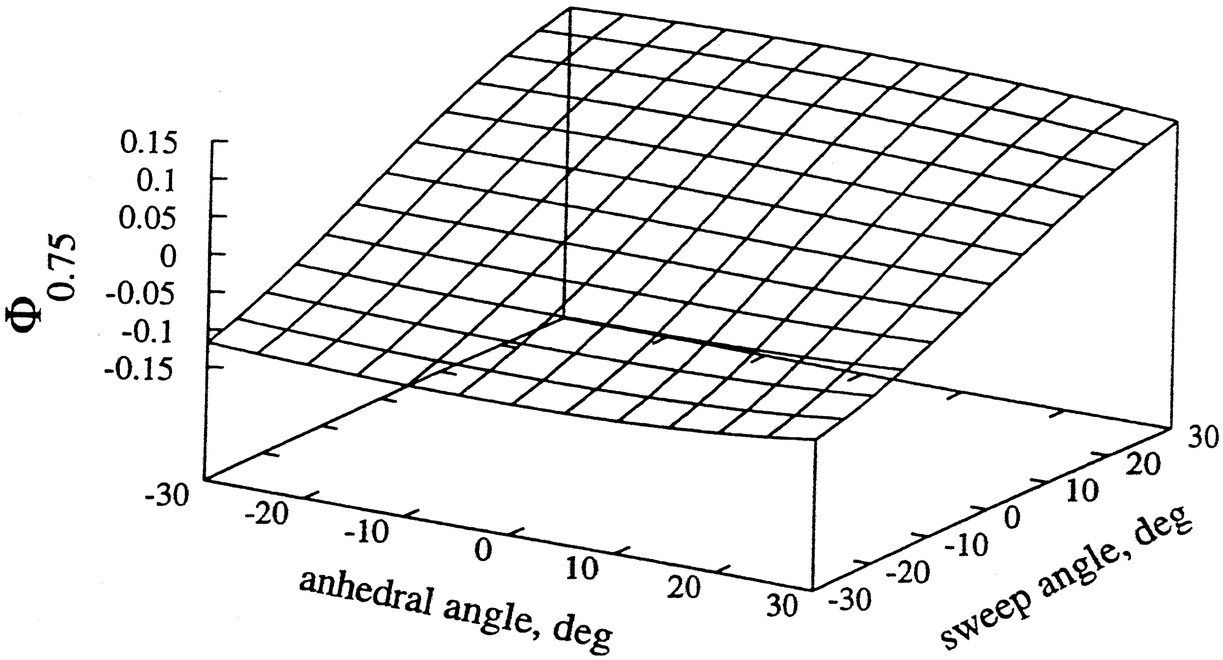


Figure 28: Flap-torsion coupling Vs tip sweep angle in first flap mode

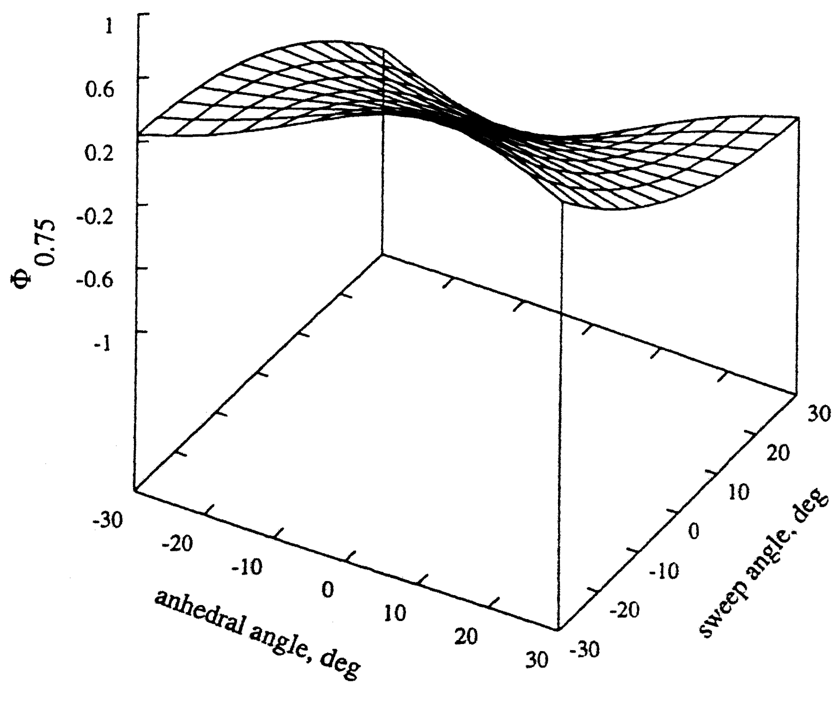


Figure 29: Flap-torsion coupling Vs tip sweep angle in second flap mode

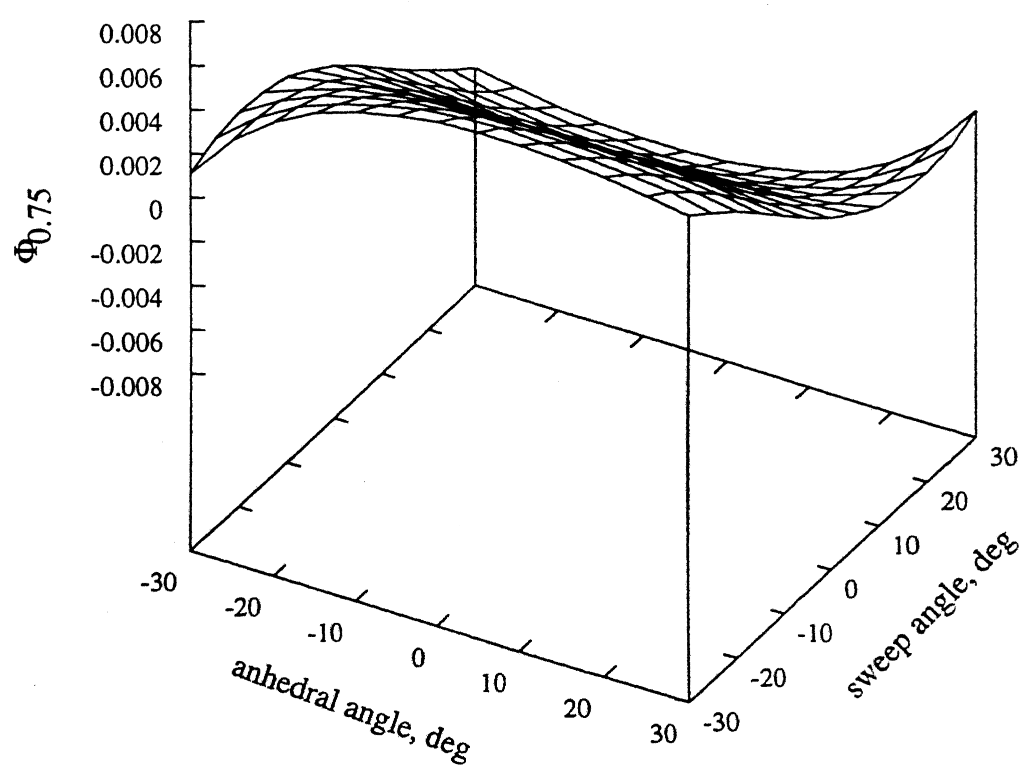


Figure 30: Flap-torsion coupling Vs tip sweep angle in thrid flap mode

References

[1] Houbolt, J.C., and Brooks, G.W., "Differential Equations of Motion for Combined Flapwise Bending, Chordwise Bending and Torsion of Twisted Nonuniform Rotor Blades," NACA Report 1346, 1958.

[2] Hodges, D.H. and Dowell, E.H., "Nonlinear Equations of Motion for the Elastic Bending and Torsion of Twisted Nonuniform Rotor Blades," NASA TN D-7818, December 1974.

[3] Rosen, A., and Friedmann, P.P., "Nonlinear Equations of Equilibrium for Elastic Helicopter or Wind Turbine Blades Undergoing Moderate Deflection," NASA CR-159478, October 1978.

[4] Kaza, K.R. and Kvaternik, R.G., "Nonlinear Aeroelastic Equations for Combined Flapwise Bending, Chordwise Bending, Torsion and Extension of Twisted Nonuniform Rotor Blades in Forward Flight," NASA TM-74059, 1977.

[5] Crespo DaSilva, M.R.M. and Hodges, D.H., "Nonlinear Flexure and Torsion of Rotating Beams with Application to Helicopter Blades-I. Formulation," Vertica, Vol. 10, 1986, pp. 151-169.

[6] Crespo DaSilva, M.R.M. and Hodges, D.H., "Nonlinear Flexure and Torsion of Rotating Beams with Application to Helicopter Blades-II. Results for Hover," Vertica, Vol. 10, 1986, pp. 171-186.

[7] Rosen, A. and Friedmann, P.P., "The Nonlinear Behavior of Elastic Slender Straight Beams Undergoing Small Strains and Moderate Rotations," Journal of Applied Mechanics, Vol. 46, March 1979, pp. 161-168.

[8] Hodges, D.H. and Ormiston, R.A., "Stability of Elastic Bending and Torsion of Uniform Cantilever Rotor Blades in Hover With Variable Structural Coupling," NASA TN D-8192, 1976.

[9] Shamie, J. and Friedmann, P.P., "Effect of Moderate Deflections on the Aeroelastic Stability of a Rotor Blade in Forward Flight," Proc. 3rd European Rotorcraft and Powered Lift Aircraft Forum, Aix-en-Provence, 1977, pp.24.1-24.37.

[10] Sivaneri, N.T. and Chopra, I., "Dynamic Stability of a Rotor Blade Using Finite Element Analysis," AIAA Journal, Vol. 20, No. 5, 1982, pp. 716-723.

[11] Celi, R. and Friedmann, P.P., "Rotor Blade Aeroelasticity in Forward Flight with an Implicit Aerodynamic Formulation," AIAA Journal, Vol. 26, No. 2, 1988, pp. 1425-1433.

[12] Bauchau, O.A., "A Beam Theory for Anistropic Materials," Journal of Applied Mechanics, Vol. 52, June 1985, pp. 416-422.

[13] Bauchau, O.A. and Hong, C.H., "Finite Element Approach to Rotor Blade Modelling," Journal of American Helicopter Society, Vol. 32, No. 1, Jan. 1987, pp. 60-67.

[14] Bauchau, O.A., Coffenberry, B.S. and Rehfield, L.W, "Composite Box Beam Analysis: Theory and Experiments," Journal of Reinforced Plastic and Composites, Vol. 6, 1987, pp. 25-35.

[15] Bauchau, O.A. and Hong, C.H., "Large Displacement Analysis of Naturally Curved and Twisted Composite Beams," AIAA Journal, Vol. 25, No. 11, Nov. 1987, pp. 1469-1475.

[16] Bauchau, O.A. and Hong, C.H., "Nonlinear Composite Beam Theory," Journal of Applied Mechanics, Vol. 55, March 1988, pp. 156-163.

[17] Kosmatka, J.B. and Friedmann, P.P., "Vibration Analysis of Composite Turbopropellers Using a Nonlinear Beam-Type Finite Element Approach," AIAA Journal, Vol. 27, No. 11, 1989, pp. 1606-1614.

[18] Hodges, D.H., "Review of Composite Rotor Blade Modelling," AIAA Journal, Vol. 28, No. 3, 1990, pp. 561-564.

[19] Yaun, K.A., Venkatesan, C., and Friedmann, P.P., "Structural Dynamic Model of Composite Rotor Blades Undergoing Moderate Deflections," in Recent Advances in Structural Dynamic Modelling of Composite Rotor Blades and Thick Composites, ASME AD-Vol. 30, pp. 127-155, ASME Winter Annual Meeting, November 1992.

[20] Smith, E.C. and Chopra. I., "Formulation and Evaluation of an Analytical Model for Composite Box Beams," Journal of the American Helicopter Society, Vol. 36, No. 3, 1991, pp. 23-35.

[21] Celi, R. and Friedmann, P.P., "Aeroelastic Modeling of Swept Tip Rotor Blades Using Finite Elements," Journal of the American Helicopter Society, Vol. 33, No. 2, April 1988, pp. 43-52.

[22] Celi, R. and Friedmann, P.P., "Structural Optimization with Aeroelastic Constraints of Rotor Blades with Straight and Swept tips," AIAA Journal, Vol. 28, No. 5, 1990, pp. 928-936.

[23] Kim, K.C., and Chopra, I., "Aeroelastic Analysis of Swept, Anhedral, and Tapered Tip Rotor Blades," Journal of the American Helicopte Society, Vol. 37, No. 1, 1992, pp. 15-30.

[24] Yuan, K.A., Friedmann, P.P. and Venkatesan, C., "A New Aeroelastic Model for Composite Rotor Blades with Straight and Swept Tips," AIAA Paper No. 92-2259, Proc. 33rd AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics and Materials Cong., Dallas, TX, April 1992, pp. 1371-1390.

[25] Yen, J.G., "Effects of Blade Tip on Dynamics, Cost, Weight, Aerodynamic Performance and Aeroelastic Response," Journal of the American Helicopter Society, Vol. 39, No. 4, Oct. 1994, pp. 37-45.

[26] Gupta, P.K., "Structural Dynamics of Helicopter Rotor having Precone - Presweep - Predroop- Pretwist and Torque offset including Hub Motions", M.Tech Thesis, Dept. of Aerospace Engineering. I.I.T Kanpur, July 1996.

[27] Pohit, G., "Dynamics of a Bearingless Helicopter Rotor Blade with a Non-linear Elastomatic Constraint," Ph.D Thesis, Dept. of Mechanical Engineering. I.I.T Kanpur, Feb. 1999.

[28] Yuan, K.A and Friedmann, P:P., "Aeroelastic and Structural Optimization of Composite Helicopter Rotor Blades With Swept Tips," NASA Contractor Report 4665, May 1995.

[29] Venkatesan. C., "Treatment of Axial Mode in Structural Dynamics and Aeroelastic Analysis of Rotor Blade -A Review," Computational Structural Mechanics, Proceedings of the NASAS-94, I.I.T. Kharagpur, Dec. 8-9, 1994, pp. 361-372.

[30] Panda. B., "Technical Note: Assembly of Moderate-Rotation Finite Elements Used in Helicopter Rotor Dynamics," Journal of the American Helicopter Society, Vol. 32, No. 4, 1987, pp. 63-69.

Appendix A

Local to global coordinate transformation for tip element

The linear transformation matrix to modify the local coordinate system of the tip element to global system is given by the expression.

$$\{q\,G\} = [\Lambda^L]\{q^t\}$$

where $\{q^t\}$ refer to the nodal degrees of freedom of the tip element in element coordinate system and $\{q^G\}$ corresponds to the global coordinate system. The non zero elements of $[\Lambda^L]$ are given below.

$$\Lambda^L(1,1) = \cos\Lambda_s \qquad \Lambda^L(1,5) = \sin\Lambda_s\ \sin\Lambda_a$$
$$\Lambda^L(1,12) = \sin\Lambda_s\ \cos\Lambda_a \qquad \Lambda^L(2,2) = \cos\Lambda_a$$
$$\Lambda^L(2,9) = -\sin\Lambda_a \qquad \Lambda^L(3,3) = \cos\Lambda_s$$
$$\Lambda^L(3,7) = \sin\Lambda_s\ \sin\Lambda_a \qquad \Lambda^L(3,14) = \sin\Lambda_s\ \cos\Lambda_a$$
$$\Lambda^L(4,4) = \cos\Lambda_a \qquad \Lambda^L(4,11) = -\sin\Lambda_a$$
$$\Lambda^L(5,5) = \cos\Lambda_a \qquad \Lambda^L(5,12) = -\sin\Lambda_a$$
$$\Lambda^L(6,2) = -\sin\Lambda_s\ \sin\Lambda_a \qquad \Lambda^L(6,6) = \cos\Lambda_s$$
$$\Lambda^L(6,9) = -\sin\Lambda_s\ \cos\Lambda_a \qquad \Lambda^L(7,7) = \cos\Lambda_a$$
$$\Lambda^L(7,14) = -\sin\Lambda_a \qquad \Lambda^L(8,4) = -\sin\Lambda_s\ \sin\Lambda_a$$
$$\Lambda^L(8,6) = \cos\Lambda_s \qquad \Lambda^L(8,11) = -\sin\Lambda_s\ \cos\Lambda_a$$
$$\Lambda^L(9,2) = \cos\Lambda_s\ \sin\Lambda_a \qquad \Lambda^L(9,6) = \sin\Lambda_s$$
$$\Lambda^L(9,9) = \cos\Lambda_s\ \cos\Lambda_a \qquad \Lambda^L(10,2) = \tfrac{1}{2}\cos\Lambda_s\ \sin\Lambda_a$$
$$\Lambda^L(10,4) = \tfrac{1}{2}\cos\Lambda_s\ \sin\Lambda_a \qquad \Lambda^L(10,6) = \tfrac{1}{2}\sin\Lambda_s$$
$$\Lambda^L(10,8) = \tfrac{1}{2}\sin\Lambda_s \qquad \Lambda^L(10,10) = \cos\Lambda_s\ \cos\Lambda_a$$
$$\Lambda^L(11,4) = \cos\Lambda_s\ \sin\Lambda_a \qquad \Lambda^L(11,8) = \sin\Lambda_s$$
$$\Lambda^L(11,11) = \cos\Lambda_s\ \cos\Lambda_a \qquad \Lambda^L(12,1) = -\sin\Lambda_s$$
$$\Lambda^L(12,5) = \cos\Lambda_s\ \sin\Lambda_a \qquad \Lambda^L(12,12) = \cos\Lambda_s\ \cos\Lambda_a$$
$$\Lambda^L(13,1) = -\tfrac{1}{2}\sin\Lambda_s \qquad \Lambda^L(13,3) = -\tfrac{1}{2}\sin\Lambda_s$$
$$\Lambda^L(13,5) = \tfrac{1}{2}\cos\Lambda_s\ \sin\Lambda_a \qquad \Lambda^L(13,7) = \tfrac{1}{2}\cos\Lambda_s\ \sin\Lambda_a$$
$$\Lambda^L(13,13) = \cos\Lambda_s\ \cos\Lambda_a \qquad \Lambda^L(14,3) = -\sin\Lambda_s$$
$$\Lambda^L(14,7) = \cos\Lambda_s\ \sin\Lambda_a \qquad \Lambda^L(14,14) = \cos\Lambda_s\ \cos\Lambda_a$$

Appendix B

Natural frequencies and coupling measures of rotor blade with swept tips

Table B.1: Natural frequencies

Λ_a deg	Λ_s deg	1 lag	1 flap	torsion	2 flap	2 lag	axial	3 flap
-30.	-30.	.8010	1.2015	3.2537	3.7440	4.7577	7.1196	8.2532
-30.	-25.	.7820	1.1994	3.2542	3.7163	4.6724	7.0912	8.1994
-30.	-20.	.7658	1.1961	3.2547	3.6880	4.6025	7.0745	8.1470
-30.	-15.	.7522	1.1915	3.2551	3.6594	4.5463	7.0695	8.0965
-30.	-10.	.7415	1.1858	3.2554	3.6307	4.5024	7.0762	8.0480
-30.	-5.	.7336	1.1789	3.2555	3.6022	4.4694	7.0937	8.0019
-30.	0.	.7284	1.1709	3.2556	3.5740	4.4463	7.1211	7.9581
-30.	5.	.7259	1.1619	3.2555	3.5463	4.4325	7.1572	7.9168
-30.	10.	.7259	1.1521	3.2554	3.5193	4.4272	7.2005	7.8779
-30.	15.	.7283	1.1415	3.2551	3.4929	4.4300	7.2492	7.8413
-30.	20.	.7330	1.1304	3.2547	3.4675	4.4406	7.3015	7.8069
-30.	25.	.7397	1.1188	3.2542	3.4431	4.4586	7.3556	7.7745
-30.	30.	.7481	1.1071	3.2537	3.4197	4.4838	7.4099	7.7440
-25.	-30.	.8059	1.1841	3.2564	3.6600	4.8149	7.1008	8.1052
-25.	-25.	.7866	1.1819	3.2568	3.6399	4.7156	7.0673	8.0693
-25.	-20.	.7699	1.1788	3.2572	3.6192	4.6345	7.0449	8.0336
-25.	-15.	.7558	1.1748	3.2574	3.5979	4.5691	7.0341	7.9983
-25.	-10.	.7444	1.1699	3.2576	3.5763	4.5173	7.0347	7.9638
-25.	-5.	.7355	1.1641	3.2578	3.5545	4.4775	7.0463	7.9302
-25.	0.	.7292	1.1575	3.2578	3.5326	4.4484	7.0680	7.8977
-25.	5.	.7254	1.1502	3.2578	3.5109	4.4288	7.0990	7.8665
-25.	10.	.7238	1.1422	3.2576	3.4894	4.4179	7.1379	7.8365
-25.	15.	.7244	1.1337	3.2574	3.4683	4.4150	7.1833	7.8078
-25.	20.	.7270	1.1248	3.2572	3.4476	4.4196	7.2335	7.7804
-25.	25.	.7315	1.1156	3.2568	3.4275	4.4312	7.2870	7.7543
-25.	30.	.7375	1.1063	3.2564	3.4080	4.4495	7.3421	7.7295

-20.	-30.	.8113	1.1686	3.2588	3.5913	4.8743	7.0884	7.9936
-20.	-25.	.7915	1.1665	3.2590	3.5775	4.7598	7.0502	7.9704
-20.	-20.	.7741	1.1637	3.2593	3.5629	4.6666	7.0226	7.9466
-20.	-15.	.7594	1.1603	3.2595	3.5477	4.5915	7.0062	7.9227
-20.	-10.	.7471	1.1562	3.2596	3.5320	4.5318	7.0010	7.8987
-20.	-5.	.7373	1.1515	3.2597	3.5159	4.4853	7.0068	7.8748
-20.	0.	.7299	1.1463	3.2597	3.4995	4.4501	7.0230	7.8512
-20.	5.	.7248	1.1406	3.2597	3.4829	4.4250	7.0488	7.8281
-20.	10.	.7217	1.1344	3.2596	3.4663	4.4087	7.0832	7.8054
-20.	15.	.7207	1.1278	3.2595	3.4498	4.4006	7.1251	7.7834
-20.	20.	.7215	1.1210	3.2593	3.4334	4.3998	7.1730	7.7620
-20.	25.	.7240	1.1140	3.2590	3.4172	4.4058	7.2254	7.7413
-20.	30.	.7279	1.1068	3.2588	3.4014	4.4180	7.2809	7.7213
-15.	-30.	.8172	1.1549	3.2607	3.5360	4.9354	7.0832	7.9092
-15.	-25.	.7966	1.1530	3.2609	3.5273	4.8044	7.0408	7.8952
-15.	-20.	.7785	1.1507	3.2610	3.5179	4.6986	7.0084	7.8804
-15.	-15.	.7629	1.1480	3.2611	3.5078	4.6135	6.9868	7.8650
-15.	-10.	.7497	1.1449	3.2612	3.4971	4.5457	6.9761	7.8492
-15.	-5.	.7390	1.1413	3.2612	3.4858	4.4924	6.9762	7.8331
-15.	0.	.7304	1.1374	3.2612	3.4742	4.4514	6.9869	7.8168
-15.	5.	.7241	1.1332	3.2612	3.4622	4.4210	7.0076	7.8003
-15.	10.	.7197	1.1287	3.2612	3.4500	4.3999	7.0375	7.7839
-15.	15.	.7172	1.1239	3.2611	3.4376	4.3870	7.0756	7.7676
-15.	20.	.7164	1.1190	3.2610	3.4251	4.3813	7.1209	7.7515
-15.	25.	.7171	1.1139	3.2609	3.4125	4.3823	7.1719	7.7356
-15.	30.	.7193	1.1088	3.2607	3.4000	4.3892	7.2273	7.7199
-10.	-30.	.8237	1.1430	3.2621	3.4921	4.9976	7.0858	7.8455
-10.	-25.	.8020	1.1416	3.2622	3.4878	4.8491	7.0398	7.8385
-10.	-20.	.7829	1.1400	3.2623	3.4827	4.7300	7.0031	7.8307
-10.	-15.	.7663	1.1381	3.2623	3.4770	4.6348	6.9766	7.8221
-10.	-10.	.7522	1.1360	3.2623	3.4707	4.5589	6.9606	7.8130
-10.	-5.	.7404	1.1336	3.2624	3.4638	4.4990	6.9553	7.8033
-10.	0.	.7308	1.1310	3.2624	3.4564	4.4524	6.9606	7.7931
-10.	5.	.7233	1.1282	3.2624	3.4486	4.4171	6.9762	7.7825
-10.	10.	.7178	1.1252	3.2623	3.4403	4.3914	7.0014	7.7715
-10.	15.	.7140	1.1221	3.2623	3.4318	4.3741	7.0356	7.7604
-10.	20.	.7118	1.1189	3.2623	3.4229	4.3642	7.0779	7.7490
-10.	25.	.7111	1.1156	3.2622	3.4138	4.3608	7.1270	7.7375
-10.	30.	.7117	1.1123	3.2621	3.4045	4.3632	7.1817	7.7260

-5.	-30.	.8308	1.1329	3.2630	3.4581	5.0599	7.0968	7.7980
-5.	-25.	.8077	1.1322	3.2630	3.4576	4.8932	7.0476	7.7966
-5.	-20.	.7874	1.1315	3.2630	3.4564	4.7606	7.0071	7.7945
-5.	-15.	.7697	1.1306	3.2630	3.4547	4.6550	6.9761	7.7916
-5.	-10.	.7545	1.1295	3.2630	3.4523	4.5712	6.9551	7.7881
-5.	-5.	.7417	1.1284	3.2631	3.4494	4.5049	6.9446	7.7840
-5.	0.	.7311	1.1271	3.2631	3.4459	4.4530	6.9446	7.7792
-5.	5.	.7225	1.1257	3.2631	3.4419	4.4130	6.9550	7.7739
-5.	10.	.7159	1.1242	3.2630	3.4374	4.3832	6.9755	7.7680
-5.	15.	.7110	1.1226	3.2630	3.4325	4.3620	7.0056	7.7616
-5.	20.	.7078	1.1210	3.2630	3.4271	4.3483	7.0445	7.7549
-5.	25.	.7059	1.1192	3.2630	3.4213	4.3412	7.0912	7.7477
-5.	30.	.7053	1.1174	3.2630	3.4152	4.3397	7.1447	7.7402
0.	-30.	.8384	1.1243	3.2633	3.4328	5.1216	7.1163	7.7636
0.	-25.	.8137	1.1248	3.2633	3.4357	4.9361	7.0647	7.7669
0.	-20.	.7919	1.1251	3.2633	3.4381	4.7898	7.0209	7.7697
0.	-15.	.7729	1.1254	3.2633	3.4400	4.6741	6.9857	7.7718
0.	-10.	.7566	1.1256	3.2633	3.4413	4.5825	6.9601	7.7734
0.	-5.	.7427	1.1257	3.2633	3.4421	4.5101	6.9445	7.7743
0.	0.	.7311	1.1258	3.2633	3.4424	4.4532	6.9393	7.7746
0.	5.	.7217	1.1257	3.2633	3.4421	4.4090	6.9445	7.7743
0.	10.	.7142	1.1256	3.2633	3.4413	4.3754	6.9601	7.7734
0.	15.	.7084	1.1254	3.2633	3.4400	4.3508	6.9857	7.7718
0.	20.	.7042	1.1251	3.2633	3.4381	4.3338	7.0209	7.7697
0.	25.	.7015	1.1248	3.2633	3.4357	4.3234	7.0647	7.7669
0.	30.	.7001	1.1243	3.2633	3.4328	4.3186	7.1163	7.7636
5.	-30.	.8466	1.1172	3.2630	3.4152	5.1815	7.1447	7.7401
5.	-25.	.8199	1.1191	3.2630	3.4213	4.9771	7.0912	7.7477
5.	-20.	.7965	1.1209	3.2630	3.4271	4.8173	7.0445	7.7549
5.	-15.	.7761	1.1226	3.2630	3.4325	4.6917	7.0056	7.7616
5.	-10.	.7585	1.1242	3.2630	3.4374	4.5927	6.9755	7.7680
5.	-5.	.7436	1.1257	3.2631	3.4419	4.5145	6.9550	7.7739
5.	0.	.7311	1.1271	3.2631	3.4459	4.4530	6.9446	7.7792
5.	5.	.7208	1.1284	3.2631	3.4494	4.4049	6.9446	7.7840
5.	10.	.7125	1.1295	3.2630	3.4523	4.3680	6.9551	7.7881
5.	15.	.7061	1.1306	3.2630	3.4547	4.3403	6.9761	7.7916
5.	20.	.7013	1.1315	3.2630	3.4564	4.3205	7.0071	7.7945
5.	25.	.6980	1.1323	3.2630	3.4576	4.3073	7.0476	7.7966
5.	30.	.6959	1.1329	3.2630	3.4581	4.2999	7.0968	7.7980

10.	-30.	.8551	1.1115	3.2621	3.4045	5.2383	7.1817	7.7260
10.	-25.	.8262	1.1152	3.2622	3.4138	5.0155	7.1270	7.7375
10.	-20.	.8009	1.1187	3.2623	3.4229	4.8425	7.0779	7.7490
10.	-15.	.7790	1.1221	3.2623	3.4318	4.7075	7.0356	7.7604
10.	-10.	.7602	1.1252	3.2623	3.4403	4.6015	7.0014	7.7715
10.	-5.	.7442	1.1282	3.2624	3.4486	4.5181	6.9762	7.7825
10.	0.	.7308	1.1310	3.2624	3.4564	4.4524	6.9606	7.7931
10.	5.	.7198	1.1336	3.2624	3.4638	4.4009	6.9553	7.8033
10.	10.	.7110	1.1360	3.2623	3.4707	4.3611	6.9606	7.8130
10.	15.	.7040	1.1382	3.2623	3.4770	4.3308	6.9766	7.8221
10.	20.	.6988	1.1401	3.2623	3.4827	4.3085	7.0031	7.8306
10.	25.	.6952	1.1419	3.2622	3.4878	4.2931	7.0398	7.8384
10.	30.	.6929	1.1434	3.2621	3.4921	4.2833	7.0858	7.8454
15.	-30.	.8639	1.1071	3.2607	3.4001	5.2908	7.2272	7.7201
15.	-25.	.8325	1.1130	3.2609	3.4126	5.0503	7.1719	7.7357
15.	-20.	.8052	1.1185	3.2610	3.4251	4.8651	7.1209	7.7515
15.	-15.	.7817	1.1237	3.2611	3.4376	4.7214	7.0756	7.7676
15.	-10.	.7616	1.1286	3.2612	3.4500	4.6091	7.0375	7.7839
15.	-5.	.7446	1.1332	3.2612	3.4622	4.5208	7.0076	7.8003
15.	0.	.7304	1.1374	3.2612	3.4742	4.4514	6.9869	7.8168
15.	5.	.7188	1.1413	3.2612	3.4858	4.3970	6.9762	7.8331
15.	10.	.7095	1.1449	3.2612	3.4971	4.3546	6.9761	7.8492
15.	15.	.7023	1.1481	3.2611	3.5078	4.3221	6.9868	7.8650
15.	20.	.6969	1.1510	3.2610	3.5179	4.2978	7.0084	7.8804
15.	25.	.6932	1.1535	3.2609	3.5273	4.2805	7.0408	7.8951
15.	30.	.6908	1.1557	3.2607	3.5359	4.2689	7.0832	7.9091
20.	-30.	.8727	1.1038	3.2588	3.4014	5.3377	7.2807	7.7218
20.	-25.	.8387	1.1123	3.2590	3.4173	5.0810	7.2253	7.7416
20.	-20.	.8093	1.1201	3.2593	3.4334	4.8846	7.1729	7.7621
20.	-15.	.7841	1.1275	3.2595	3.4498	4.7330	7.1251	7.7834
20.	-10.	.7628	1.1343	3.2596	3.4663	4.6151	7.0832	7.8054
20.	-5.	.7448	1.1405	3.2597	3.4829	4.5227	7.0488	7.8281
20.	0.	.7299	1.1463	3.2597	3.4995	4.4501	7.0230	7.8512
20.	5.	.7178	1.1515	3.2597	3.5159	4.3931	7.0068	7.8748
20.	10.	.7082	1.1563	3.2596	3.5320	4.3485	7.0010	7.8987
20.	15.	.7008	1.1604	3.2595	3.5477	4.3142	7.0062	7.9227
20.	20.	.6955	1.1641	3.2593	3.5629	4.2883	7.0226	7.9466
20.	25.	.6919	1.1673	3.2590	3.5775	4.2695	7.0502	7.9703
20.	30.	.6898	1.1699	3.2588	3.5912	4.2565	7.0884	7.9935

25.	-30.	.8814	1.1015	3.2564	3.4082	5.3776	7.3415	7.7305
25.	-25.	.8446	1.1130	3.2568	3.4276	5.1067	7.2866	7.7550
25.	-20.	.8130	1.1235	3.2572	3.4477	4.9005	7.2333	7.7808
25.	-15.	.7862	1.1332	3.2574	3.4683	4.7422	7.1832	7.8080
25.	-10.	.7636	1.1420	3.2576	3.4894	4.6195	7.1379	7.8365
25.	-5.	.7447	1.1502	3.2578	3.5109	4.5237	7.0990	7.8665
25.	0.	.7292	1.1575	3.2578	3.5326	4.4484	7.0680	7.8977
25.	5.	.7168	1.1641	3.2578	3.5545	4.3893	7.0463	7.9302
25.	10.	.7070	1.1699	3.2576	3.5763	4.3430	7.0347	7.9638
25.	15.	.6997	1.1750	3.2574	3.5979	4.3072	7.0341	7.9984
25.	20.	.6945	1.1794	3.2572	3.6192	4.2801	7.0449	8.0336
25.	25.	.6912	1.1830	3.2568	3.6399	4.2600	7.0673	8.0693
25.	30.	.6896	1.1859	3.2564	3.6599	4.2460	7.1008	8.1051
30.	-30.	.8898	1.1003	3.2537	3.4199	5.4094	7.4085	7.7461
30.	-25.	.8500	1.1152	3.2542	3.4433	5.1268	7.3548	7.7758
30.	-20.	.8164	1.1285	3.2547	3.4676	4.9126	7.3011	7.8076
30.	-15.	.7879	1.1407	3.2551	3.4930	4.7489	7.2490	7.8416
30.	-10.	.7641	1.1518	3.2554	3.5193	4.6223	7.2004	7.8780
30.	-5.	.7444	1.1618	3.2555	3.5463	4.5237	7.1572	7.9169
30.	0.	.7284	1.1709	3.2556	3.5740	4.4463	7.1211	7.9581
30.	5.	.7157	1.1788	3.2555	3.6022	4.3856	7.0937	8.0019
30.	10.	.7059	1.1858	3.2554	3.6308	4.3380	7.0761	8.0481
30.	15.	.6988	1.1917	3.2551	3.6595	4.3011	7.0695	8.0965
30.	20.	.6940	1.1966	3.2547	3.6880	4.2730	7.0744	8.1471
30.	25.	.6912	1.2006	3.2542	3.7163	4.2521	7.0912	8.1995
30.	30.	.6902	1.2036	3.2537	3.7439	4.2374	7.1196	8.2532

Table B.2: The variation of torsion coupling ($\Phi_{0.75}$) as a function of sweep angles for the lag and flap modes

Λ_a deg	Λ_s deg	1 lag	1 flap	torsion	2 flap	2 lag	3 flap
-30.	-30.	.1003	-.1169	.9198	.2421	-.0570	.0011
-30.	-25.	.1046	-.1034	.9198	.2273	-.0692	.0017
-30.	-20.	.1087	-.0868	.9199	.2034	-.0795	.0019
-30.	-15.	.1123	-.0674	.9199	.1695	-.0873	.0018
-30.	-10.	.1152	-.0461	.9200	.1249	-.0926	.0015
-30.	-5.	.1170	-.0233	.9200	.0686	-.0958	.0008
-30.	0.	.1176	.0000	.9200	.0000	-.0978	.0000
-30.	5.	.1168	.0232	.9200	-.0817	-.0990	-.0010
-30.	10.	.1144	.0456	.9200	-.1773	-.1000	-.0021
-30.	15.	.1104	.0664	.9199	-.2874	-.1011	-.0032
-30.	20.	.1045	.0851	.9199	-.4125	-.1021	-.0042
-30.	25.	.0968	.1010	.9198	-.5534	-.1026	-.0052
-30.	30.	.0873	.1137	.9198	-.7108	-.1024	-.0061
-25.	-30.	.0906	-.1200	.9201	.3080	-.0459	.0032
-25.	-25.	.0936	-.1062	.9201	.2869	-.0573	.0032
-25.	-20.	.0967	-.0890	.9202	.2547	-.0673	.0030
-25.	-15.	.0995	-.0692	.9202	.2106	-.0752	.0025
-25.	-10.	.1019	-.0472	.9202	.1539	-.0807	.0019
-25.	-5.	.1034	-.0239	.9203	.0839	-.0843	.0010
-25.	0.	.1039	.0000	.9203	.0000	-.0864	.0000
-25.	5.	.1032	.0238	.9203	-.0982	-.0879	-.0011
-25.	10.	.1010	.0468	.9202	-.2110	-.0894	-.0023
-25.	15.	.0973	.0682	.9202	-.3388	-.0913	-.0034
-25.	20.	.0920	.0874	.9202	-.4817	-.0935	-.0046
-25.	25.	.0849	.1038	.9201	-.6399	-.0957	-.0056
-25.	30.	.0762	.1170	.9201	-.8134	-.0975	-.0064

-20.	-30.	.0783	-.1225	.9204	.3873	-.0342	.0046
-20.	-25.	.0798	-.1083	.9204	.3576	-.0445	.0043
-20.	-20.	.0818	-.0908	.9204	.3145	-.0538	.0038
-20.	-15.	.0838	-.0706	.9205	.2576	-.0614	.0031
-20.	-10.	.0855	-.0482	.9205	.1864	-.0668	.0022
-20.	-5.	.0867	-.0244	.9205	.1005	-.0703	.0011
-20.	0.	.0871	.0000	.9205	.0000	-.0725	.0000
-20.	5.	.0864	.0243	.9205	-.1152	-.0741	-.0012
-20.	10.	.0846	.0479	.9205	-.2449	-.0759	-.0024
-20.	15.	.0813	.0698	.9205	-.3886	-.0784	-.0037
-20.	20.	.0766	.0895	.9204	-.5456	-.0816	-.0048
-20.	25.	.0704	.1064	.9204	-.7151	-.0855	-.0058
-20.	30.	.0628	.1199	.9204	-.8959	-.0893	-.0067
-15.	-30.	.0638	-.1244	.9206	.4802	-.0223	.0056
-15.	-25.	.0638	-.1100	.9206	.4386	-.0312	.0050
-15.	-20.	.0644	-.0922	.9206	.3816	-.0394	.0043
-15.	-15.	.0655	-.0717	.9206	.3090	-.0462	.0034
-15.	-10.	.0666	-.0490	.9207	.2209	-.0512	.0024
-15.	-5.	.0674	-.0248	.9207	.1177	-.0544	.0012
-15.	0.	.0677	.0000	.9207	.0000	-.0564	.0000
-15.	5.	.0671	.0248	.9207	-.1314	-.0579	-.0013
-15.	10.	.0656	.0487	.9207	-.2753	-.0598	-.0026
-15.	15.	.0629	.0711	.9206	-.4302	-.0627	-.0038
-15.	20.	.0590	.0913	.9206	-.5945	-.0669	-.0050
-15.	25.	.0538	.1085	.9206	-.7658	-.0722	-.0060
-15.	30.	.0473	.1225	.9206	-.9417	-.0780	-.0069
-10.	-30.	.0477	-.1258	.9208	.5848	-.0106	.0062
-10.	-25.	.0459	-.1113	.9208	.5272	-.0176	.0055
-10.	-20.	.0452	-.0933	.9208	.4525	-.0244	.0047
-10.	-15.	.0452	-.0726	.9208	.3613	-.0301	.0037
-10.	-10.	.0457	-.0496	.9208	.2546	-.0343	.0025
-10.	-5.	.0461	-.0252	.9208	.1336	-.0370	.0013
-10.	0.	.0462	.0000	.9208	.0000	-.0385	.0000
-10.	5.	.0459	.0251	.9208	-.1444	-.0398	-.0013
-10.	10.	.0447	.0494	.9208	-.2974	-.0417	-.0027
-10.	15.	.0427	.0722	.9208	-.4565	-.0448	-.0039
-10.	20.	.0396	.0927	.9208	-.6188	-.0497	-.0051
-10.	25.	.0354	.1103	.9208	-.7810	-.0562	-.0061
-10.	30.	.0303	.1245	.9208	-.9395	-.0639	-.0070

-5.	-30.	.0303	-.1266	.9209	.6958	.0006	.0066
-5.	-25.	.0267	-.1120	.9209	.6171	-.0043	.0059
-5.	-20.	.0246	-.0940	.9209	.5209	-.0093	.0049
-5.	-15.	.0237	-.0731	.9209	.4089	-.0136	.0038
-5.	-10.	.0234	-.0500	.9209	.2832	-.0168	.0026
-5.	-5.	.0234	-.0254	.9209	.1460	-.0186	.0013
-5.	0.	.0235	.0000	.9209	.0000	-.0196	.0000
-5.	5.	.0233	.0254	.9209	-.1520	-.0204	-.0014
-5.	10.	.0225	.0499	.9209	-.3070	-.0219	-.0027
-5.	15.	.0212	.0729	.9209	-.4618	-.0251	-.0040
-5.	20.	.0190	.0937	.9209	-.6129	-.0305	-.0051
-5.	25.	.0160	.1116	.9209	-.7566	-.0381	-.0062
-5.	30.	.0123	.1260	.9209	-.8893	-.0475	-.0070
0.	-30.	.0123	-.1267	.9209	.8025	.0111	.0069
0.	-25.	.0068	-.1122	.9209	.6980	.0085	.0061
0.	-20.	.0033	-.0941	.9209	.5778	.0055	.0051
0.	-15.	.0014	-.0732	.9209	.4447	.0028	.0039
0.	-10.	.0004	-.0501	.9209	.3019	.0010	.0027
0.	-5.	.0000	-.0254	.9209	.1526	.0001	.0014
0.	0.	.0000	.0000	.9209	.0000	.0000	.0000
0.	5.	.0000	.0254	.9209	-.1526	-.0002	-.0014
0.	10.	-.0003	.0501	.9209	-.3019	-.0013	-.0027
0.	15.	-.0010	.0732	.9209	-.4447	-.0043	-.0039
0.	20.	-.0022	.0941	.9209	-.5778	-.0099	-.0051
0.	25.	-.0039	.1122	.9209	-.6980	-.0183	-.0061
0.	30.	-.0062	.1267	.9209	-.8025	-.0292	-.0069
5.	-30.	-.0059	-.1259	.9209	.8893	.0206	.0070
5.	-25.	-.0132	-.1115	.9209	.7566	.0204	.0062
5.	-20.	-.0180	-.0936	.9209	.6128	.0196	.0051
5.	-15.	-.0210	-.0729	.9209	.4618	.0187	.0040
5.	-10.	-.0226	-.0499	.9209	.3070	.0184	.0027
5.	-5.	-.0233	-.0253	.9209	.1520	.0187	.0014
5.	0.	-.0235	.0000	.9209	.0000	.0196	.0000
5.	5.	-.0233	.0254	.9209	-.1460	.0202	-.0013
5.	10.	-.0231	.0500	.9209	-.2832	.0197	-.0026
5.	15.	-.0231	.0731	.9209	-.4089	.0170	-.0038
5.	20.	-.0233	.0940	.9209	-.5209	.0114	-.0049
5.	25.	-.0238	.1120	.9209	-.6171	.0024	-.0059
5.	30.	-.0246	.1266	.9209	-.6958	-.0097	-.0066

10.	-30.	-.0236	-.1239	.9208	.9394	.0290	.0070
10.	-25.	-.0328	-.1098	.9208	.7810	.0312	.0061
10.	-20.	-.0389	-.0923	.9208	.6188	.0326	.0051
10.	-15.	-.0427	-.0719	.9208	.4565	.0337	.0039
10.	-10.	-.0450	-.0492	.9208	.2974	.0350	.0027
10.	-5.	-.0460	-.0250	.9208	.1444	.0367	.0013
10.	0.	-.0462	.0000	.9208	.0000	.0385	.0000
10.	5.	-.0459	.0251	.9208	-.1336	.0401	-.0013
10.	10.	-.0453	.0494	.9208	-.2546	.0403	-.0025
10.	15.	-.0445	.0723	.9208	-.3613	.0382	-.0037
10.	20.	-.0437	.0930	.9208	-.4525	.0327	-.0047
10.	25.	-.0430	.1109	.9208	-.5273	.0235	-.0055
10.	30.	-.0423	.1255	.9208	-.5849	.0104	-.0062
15.	-30.	-.0405	-.1203	.9206	.9415	.0361	.0069
15.	-25.	-.0513	-.1068	.9206	.7658	.0408	.0060
15.	-20.	-.0585	-.0899	.9206	.5945	.0444	.0050
15.	-15.	-.0632	-.0701	.9206	.4303	.0475	.0038
15.	-10.	-.0660	-.0481	.9207	.2753	.0504	.0026
15.	-5.	-.0674	-.0245	.9207	.1314	.0534	.0013
15.	0.	-.0677	.0000	.9207	.0000	.0564	.0000
15.	5.	-.0672	.0245	.9207	-.1177	.0588	-.0012
15.	10.	-.0661	.0484	.9207	-.2209	.0599	-.0024
15.	15.	-.0646	.0709	.9206	-.3090	.0584	-.0034
15.	20.	-.0629	.0913	.9206	-.3817	.0534	-.0043
15.	25.	-.0610	.1089	.9206	-.4387	.0441	-.0050
15.	30.	-.0589	.1233	.9206	-.4803	.0304	-.0056
20.	-30.	-.0559	-.1150	.9204	.8956	.0420	.0067
20.	-25.	-.0682	-.1025	.9204	.7149	.0489	.0058
20.	-20.	-.0764	-.0864	.9204	.5456	.0547	.0048
20.	-15.	-.0818	-.0675	.9205	.3886	.0597	.0037
20.	-10.	-.0851	-.0464	.9205	.2450	.0642	.0025
20.	-5.	-.0867	-.0236	.9205	.1153	.0685	.0012
20.	0.	-.0871	.0000	.9205	.0000	.0725	.0000
20.	5.	-.0864	.0238	.9205	-.1006	.0759	-.0011
20.	10.	-.0849	.0470	.9205	-.1865	.0778	-.0022
20.	15.	-.0828	.0688	.9205	-.2578	.0771	-.0031
20.	20.	-.0802	.0887	.9204	-.3147	.0728	-.0038
20.	25.	-.0772	.1060	.9204	-.3578	.0638	-.0043
20.	30.	-.0738	.1200	.9204	-.3876	.0497	-.0046

25.	-30.	-.0695	-.1079	.9201	.8130	.0466	.0064
25.	-25.	-.0831	-.0967	.9201	.6398	.0555	.0056
25.	-20.	-.0920	-.0819	.9202	.4818	.0632	.0046
25.	-15.	-.0979	-.0642	.9202	.3389	.0700	.0035
25.	-10.	-.1016	-.0442	.9202	.2112	.0760	.0023
25.	-5.	-.1035	-.0226	.9203	.0983	.0814	.0011
25.	0.	-.1039	.0000	.9203	.0000	.0864	.0000
25.	5.	-.1031	.0228	.9203	-.0840	.0907	-.0010
25.	10.	-.1012	.0451	.9202	-.1541	.0934	-.0019
25.	15.	-.0985	.0661	.9202	-.2109	.0938	-.0025
25.	20.	-.0951	.0853	.9202	-.2551	.0903	-.0030
25.	25.	-.0911	.1020	.9201	-.2874	.0819	-.0032
25.	30.	-.0865	.1157	.9201	-.3086	.0679	-.0032
30.	-30.	-.0809	-.0988	.9198	.7103	.0499	.0061
30.	-25.	-.0953	-.0895	.9198	.5534	.0605	.0052
30.	-20.	-.1048	-.0764	.9199	.4127	.0699	.0043
30.	-15.	-.1111	-.0601	.9199	.2877	.0782	.0032
30.	-10.	-.1151	-.0415	.9200	.1776	.0855	.0021
30.	-5.	-.1172	-.0213	.9200	.0819	.0920	.0010
30.	0.	-.1176	.0000	.9200	.0000	.0978	.0000
30.	5.	-.1167	.0216	.9200	-.0688	.1027	-.0008
30.	10.	-.1145	.0427	.9200	-.1252	.1064	-.0015
30.	15.	-.1113	.0628	.9199	-.1701	.1078	-.0018
30.	20.	-.1072	.0812	.9199	-.2040	.1056	-.0019
30.	25.	-.1023	.0972	.9198	-.2281	.0983	-.0017
30.	30.	-.0968	.1104	.9198	-.2430	.0848	-.0011